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(4) Tech. Rpt. No. 4: "Sidelobe-Reduction and Interference-Suppression Techniques for Phased Arrays Using Digital Phase Shifters," RADC-TR-70-51, February 1970.

The present report is divided into two parts. Part (A) summaries the essential optimization techniques for antenna arrays, including a method which makes only phase adjustments. Part (B) presents a new integral-equation approach for optimizing arrays with mutual coupling. This approach is particularly advantageous when an array contains many long dipole elements.

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ANTENNA OPTIMIZATION CRITERIA

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FOREWORD

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This technical report has been reviewed and is approved.

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ANTENNA OPTIMIZATION CRITERIA

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PART (A). OPTIMIZATION TECHNIQUES FOR ANTENNA ARRAYS

I. INTRODUCTION

An antenna is an essential part of any electronic system which transmits or receives electromagnetic energy in a wireless fashion. Without an antenna, electromagnetic energy will be localized, and interaction at a distance between unconnected points in space does not occur. An antenna can be considered as a transducer which converts electromagnetic waves in space to current or voltage variations in a circuit, or vice versa. It must be an efficient radiator (or collector) of electromagnetic energy, and it should direct the energy to certain desired directions and suppress it in other specified directions. Thus, one must be concerned not only with the conversion efficiency of an antenna, but also with its spatial response, or radiation pattern. In an environment which does not involve nonlinear media, a reciprocity relation holds such that the properties (pattern, gain, impedance) of an antenna used for receiving are identical with those when it is transmitting. For simplicity we shall then refer all discussions to radiation properties.

A radiating element of electromagnetic energy may take many different forms. It may be a piece of conducting wire, a dielectric rod, a metallic horn, or a slot on the side of a waveguide. The radiation pattern of a single element is fixed for a given frequency of excitation and contains, in general, a main beam and a number of smaller sidelobes. In practical applications there is quite often a need for either improving the directive properties or controlling the sidelobe structure of the radiation pattern. Two methods are available for this purpose: one method is to use an appropriately shaped reflector or lens fed by a radiating element, and the other is to employ a

number of radiating elements properly arranged in space to form an antenna array. When it is necessary to steer (scan) the main beam of the radiation pattern, the requisite motion of heavy reflector-or lens-type antennas entails both mechanical and structural problems. Moreover, the possible rate of scan is severely limited. On the other hand, beam-steering for an antenna array can be accomplished electronically by adjusting the relative phase of excitation in the array elements with no need for mechanical motion, resulting in a phased array. We shall concern ourselves only with phased array antennas in this report.

The radiation pattern of an array antenna obviously depends on both the array geometry and the pattern of the individual array elements Aside from such circuit properties as impedance and efficiency, the parameters which characterize antenna performance are all based on the shape of the radiation pattern. Performance optimization then is a procedure for the maximization or minimization of certain measures on the radiation pattern. One well-known result in this regard is the Chebyshev array which makes use of the properties of Chebyshev-Akhiezer polynomials [1], [2]. A Chebyshev array is optimum in the sense that for a specified sidelobe level the width of the main beam of its radiation pattern is a minimum. Conversely, for a specified beamwidth all the sidelobes of a Chebyshev array are of equal height and are at a lowest level The original Chebyshev design considered by Dolph [1] was limited to linear broadside arrays of isotropic elements uniformly spaced at a distance equal to or larger than a half-wavelength. It has since been extended in various ways (see, for instance, [3]). Recently methods for determining the

current distribution, the minimum required number of controlled elements, and other properties of optimum rectangular arrays with a steerable main beam and constant sidelobes have been formulated [4,17].

Besides the beamwidth-sidelobe relationship, an important performance index for any antenna is its directive gain, or directivity. Directivity is defined as the ratio of the radiation intensity (radiated power per unit solid angle) in the direction of the main beam to the average radiation intensity. To put it another way, the directivity of an antenna or an array is the ratio of its maximum radiation intensity to the radiation intensity of an isotropic (omnidirectional) source* radiating the same total power. It measures the ability of concentrating the radiated energy in the main-beam direction. Our attention in this report will be directed toward the various techniques for maximizing the directivity of antenna arrays.

The performance of an antenna system as a receiving device is often constrained by the presence of a spatially distributed background noise as well as by the noise generated in the receiving system. A useful performance index of a receiving array is the signal-to-noise power ratio (SNR) at the system output. The problem of finding the complex weighting factors of the individual array elements such that the SNR is maximized for a signal coming from a given direction and a noise of a given power spectral density and a given spatial distribution is of considerable importance. It

*It should be noted that an isotropic source radiating uniformly in all directions is physically unrealizable for vector fields.

can be shown [5,6] that techniques similar to those for the maximization of directivity are applicable for SNR maximization. The formulation becomes more involved when the element space-frequency response, the signal power spectral density, the internal noise power spectral density, the spatial noise cross-power spectral density, and the receiver or filter frequency response are to be considered [7]. We shall not attempt to discuss the more general case in this report.

The problem of maximizing the directivity of a linear array with equally spaced isotropic elements was first studied by Uzkov in 1946 [8]. He demonstrated that the maximum obtainable directivity for an array with N elements spaced at a half-wavelength ($\lambda/2$) apart is N and that it tends to N^2 as the spacings approach zero. Bloch, Medhurst and Pool [9] examined the maximum directivity of a linear array of half-wave dipoles from the point of view of self and mutual resistances of the elements. Gain optimization under a specified constraint was investigated by Uzsoky and Solymar [10], and Lo, Lee and Lee [5]. In 1964 Tai [11] published many interesting curves showing the optimum directivity of various types of uniformly spaced broadside arrays, linear arrays with maximum radiation in the direction of the array normal. The optimization problem was generalized by Cheng and Tseng [12], [13] to include arrays of non-isotropic elements arranged in an arbitrary configuration with a main beam pointing at an arbitrary direction. By making use of a theorem on the properties of a ratio of two Hermitian forms in matrix algebra, the optimization procedure was formalized in a concise manner. It turned out that Krupitskii in U.S.S.R. [14] had used the same theorem to prove the existence

and uniqueness of a solution for exciting an array of discrete radiators for maximum directivity.

In the following we shall first express, in Section II, the directivity and signal-to-noise power ratio of an array of discrete elements as a ratio of two Hermitian forms. The optimization principle for a ratio of Hermitian forms is then reviewed and applied to antenna arrays in Section III. With a given array configuration where the element positions are not to be changed, the excitation amplitudes and phases in the array elements can be adjusted for the optimization of a performance index. If the array has a total of N elements, the optimization procedure involves the determination of 2N parameters. Typical results for linear and circular arrays will be presented. For linear arrays the spacings between the array elements represent another convenient set of parameters that can be adjusted to improve the performance index further. A spacing-perturbation technique which can be used in conjunction with the adjustments in excitation amplitudes and phases is discussed in Section IV. This process of excitation and spacing adjustments may be repeated until the improvement obtained by further adjustments is no longer significant.

In practice it is perhaps inconvenient to adjust the element spacings. Even adjustments in excitation amplitudes are difficult and expensive to make. Techniques for maximizing the directivity of a fixed array by keeping the amplitudes equal and adjusting the phases only are therefore of interest. These techniques are explored in Section V.

The adjustment of the excitation amplitudes, phases, or spacings of an array in order to achieve a maximum directivity changes the array radiation pattern. In practice it is often desirable to control some aspects of the array pattern. For example, one may wish to have a maximum directivity in one direction while requiring a null in certain other directions in order to minimize interference. The method of optimization under constraints is discussed in Section VI.

Initially, as we develop the optimization procedure, we shall assume that all the elements in an array are identically polarized and have the same radiation pattern (element pattern). Once the element pattern is specified, the physical structure of the array elements is no longer important in our problem, and it is immaterial whether the elements are dipoles, horns, slots, or other apertures. This assumption neglects the implications of mutual coupling. For large arrays with many elements this assumption gives acceptable results, although the element pattern must first be found. However, for arrays with a small number of closely spaced elements or in cases where more accurate results are desired, mutual-coupling effects must be taken into account. Section VII reviews the moment method for optimizing the directivity of arrays of wire antennas without neglecting mutual coupling. The moment method provides numerical solutions by first converting the governing integro-differential equations into matrix equations.

Finally, in Section VIII, we discuss various other factors which are relevant in the optimization of discrete antenna arrays.

II. GENERAL FORMULATION

Consider an array of N discrete, similarly oriented, identical elements arranged arbitrarily in a 3-dimensional space, as shown in Fig. 1. Denoting the excitation in the nth element located at (r_n,θ_n,ϕ_n) by $I_n \exp(j\psi_n)$, we may write the array factor for electric field intensity as

$$E(\theta,\phi) = \sum_{n=1}^{N} I_n \exp[j(\psi_n + kr_n \cos \alpha_n)] \tag{1}$$

where

$$\cos \alpha_n = \sin \theta \sin \theta_n \cos(\phi-\phi_n) + \cos \theta \cos \theta_n \tag{2}$$

$k = 2\pi/\lambda$ is the wavenumber, and λ is the operating wavelength. Let $g(\theta,\phi)$ denote the element power-pattern function which is normalized such that

$$g(\theta_o,\phi_o) = 1 \tag{3}$$

in the direction (θ_o,ϕ_o) of the main beam. The directivity of the array is then

$$D = \frac{\text{Radiation intensity in direction } (\theta_o,\phi_o)}{\text{Average radiation intensity}}$$

$$= \frac{|E(\theta_o,\phi_o)|^2}{\frac{1}{4\pi}\int_0^{2\pi} d\phi \int_0^{\pi} |E(\theta,\phi)|^2 g(\theta,\phi) \sin \theta d\theta} . \tag{4}$$

We now define two N-element column vectors $\underset{\sim}{J}$ and $\underset{\sim}{F}_o$, with $\underset{\sim}{J}$ representing the set of complex element excitation functions:

$$\underset{\sim}{J} = [J_n] = \begin{bmatrix} I_1 \exp(j\psi_1) \\ I_2 \exp(j\psi_2) \\ \cdot \\ \cdot \\ \cdot \\ I_N \exp(j\psi_N) \end{bmatrix} \tag{5}$$

and $\underset{\sim}{F}_o$ representing the set of phase factors due to differences in distance:

$$F_o = [F_{on}] = \begin{bmatrix} \exp(-jkr_1 \cos \alpha_{01}) \\ \exp(-jkr_2 \cos \alpha_{02}) \\ \cdot \\ \cdot \\ \cdot \\ \exp(-jkr_N \cos \alpha_{ON}) \end{bmatrix}, \tag{6}$$

where $\cos \alpha_{on}(n=1,2,\ldots,N)$ is obtained from (2) by setting $\theta = \theta_o$ and $\phi = \phi_o$. From (1), (4), (5) and (6) it is readily verified that the directivity can be written as

$$D = \frac{\underset{\sim}{J}^{\dagger} \underset{\sim}{A} \underset{\sim}{J}}{\underset{\sim}{J}^{\dagger} \underset{\sim}{B} \underset{\sim}{J}}, \tag{7}$$

where $\dagger$ on a matrix indicates the adjoint, or the conjugate transpose, of the matrix. $\underset{\sim}{A}$ and $\underset{\sim}{B}$ are N by N square matrices defined as

$$\underset{\sim}{A} = [a_{mn}] = \underset{\sim}{F}_o \underset{\sim}{F}_o^{\dagger} \tag{8}$$

and

$$\underset{\sim}{B} = [b_{mn}] \tag{9}$$

with

$$b_{mn} = \frac{1}{4\pi} \int_0^{2\pi} d\phi \int_0^{\pi} g(\theta,\phi) \exp[-jk(r_m \cos \alpha_m - r_n \cos \alpha_n)] \sin \theta d\theta \tag{10}$$

It is obvious that matrices $\underset{\sim}{A}$ and $\underset{\sim}{B}$ are both Hermitian, i.e., $\underset{\sim}{A}^{\dagger} = \underset{\sim}{A}$ ($a^*_{nm} = a_{mn}$), and $\underset{\sim}{B}^{\dagger} = \underset{\sim}{B}$ ($b^*_{nm} = b_{mn}$). Hence D in (7) is a ratio of two Hermitian forms.[1] In addition, $\underset{\sim}{B}$ is positive-definite, which implies that for any $\underset{\sim}{J} \neq \underset{\sim}{0}$, $\underset{\sim}{J}^{\dagger}\underset{\sim}{B}\,\underset{\sim}{J} > 0$. This is proved in reference [13]. Since the elements of matrices $\underset{\sim}{A}$ and $\underset{\sim}{B}$ are known when the array geometry, the operating wavelength and the scan angle are given, the optimization problem reduces to the determination of the excitation matrix $\underset{\sim}{J}$ such that D in (7) is maximized.

In receiving systems the output signal-to-noise ratio, instead of the directivity, is of interest. The <u>output SNR</u> may be defined as the ratio of the power received per unit solid angle in the direction of the signal to the average noise power received per unit solid angle. It is only necessary to replace the power-pattern function $g(\theta,\phi)$ in (4) by a more general weighting function $w(\theta,\phi)$ which includes the spatial distribution of noise power. We write

$$w(\theta,\phi) = g(\theta,\phi)\; T(\theta,\phi) \;, \tag{11}$$

where $T(\theta,\phi)$ is the spatial distribution function of noise power. Here we understand noise to be a combination of interference, clutter, atmospherics, and random noise. It is clear that replacing $g(\theta,\phi)$ by $w(\theta,\phi)$ does not change the nature of the optimization problem. In fact, the expression for SNR reduces to that for D when $T(\theta,\phi) = 1$. We expect a suppression of the sidelobes in the directions of high noise power for SNR improvement [6]. In the next section we state the theorem for maximizing D by adjusting $\underset{\sim}{J}$.

---

[1] Sometimes Hermitian forms are written as general inner products; for instance,

$$\underset{\sim}{J}^{\dagger}\underset{\sim}{A}\,\underset{\sim}{J} = \sum_{m=1}^{N}\sum_{n=1}^{N} J^*_m\, a_{mn}\, J_n = \langle \underset{\sim}{J}, \underset{\sim}{A}\,\underset{\sim}{J} \rangle \;.$$

They are quadratic forms in Hilbert space in the variables J_n.

III. OPTIMIZATION BY EXCITATION ADJUSTMENTS

A theorem in matrix algebra on the properties of a ratio of two Hermitian forms [15] is useful for the maximization of directivity by excitation adjustments. It may be stated as follows:

<u>Theorem 1</u> - If a quantity D is expressible as a ratio of two Hermitian forms as in (7) and if $\underset{\sim}{B}$ is nonsingular and positive definite, then the largest eigenvalue λ_M of the "regular pencil" of matrices $\underset{\sim}{A} - \lambda\underset{\sim}{B}$ is the maximum obtainable D when $\underset{\sim}{J}$ is the eigenvector satisfying the homogeneous equation

$$\underset{\sim}{A}\,\underset{\sim}{J} = \lambda_M\,\underset{\sim}{B}\,\underset{\sim}{J} \quad . \tag{12}$$

For our case, the following corollary, proved in [13], makes the optimization procedure particularly straightforward and simple.

<u>Corollary</u> - If A in (7) is expressible in the form of (8), then

(a) the largest and only nonzero eigenvalue of the regular pencil $\underset{\sim}{A} - \lambda\underset{\sim}{B}$ is

$$\lambda_M = D_M = \underset{\sim}{F}_o^{\dagger}\, B^{-1} F_o , \tag{13}$$

and

(b) the eigenvector corresponding to λ_M is

$$\underset{\sim}{J}_M = \underset{\sim}{B}^{-1}\underset{\sim}{F}_o \quad . \tag{14}$$

Equations (13) and (14) solve the problem of optimization by excitation adjustments for an arbitrary array. For a given array configuration, it is only necessary to determine the elements of the matrices $\underset{\sim}{F}_o$ and $\underset{\sim}{B}$, in accordance with (6) and (10) respectively.

For a linear array with elements located arbitrarily at distances d_n from a reference point,

$$a_{mn} = \exp[-jk(d_n - d_m)\sin\theta_o] \tag{15}$$

where θ_o denotes the main-beam direction measured from the normal to the array, and

$$b_{mn} = \frac{1}{4\pi}\int_0^{2\pi} d\phi \int_0^{\pi} g(\theta,\phi)\,\exp[-jk(d_n-d_m)\sin\theta]d\theta\ . \tag{16}$$

If the array elements are isotropic, $g(\theta,\phi) = 1$, and are equally spaced, $d_n-d_m = (n-m)d$, (15) and (16) reduce to

$$a_{mn} = \exp[jk(m-n)d\sin\theta_o] \tag{17}$$

and

$$b_{mn} = \frac{\sin k(m-n)d}{(m-n)d}\ . \tag{18}$$

As an example, it has been shown [12] that an endfire array with 8 isotropic elements equally spaced at 0.425λ apart has a directivity of 12.5 with a uniform amplitude and cophasal excitation.* Optimization by the above procedure results in a directivity of 22.0. The amplitude is tapered (center-to-edge ratio: 1.69) and the phase shift between adjacent elements is roughly 170°. Other examples of linear-array optimization by excitation adjustments

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*An excitation is said to be cophasal when the relative phases in the elements are adjusted such that the contributions of all the elements add in phase in the main-beam direction. Thus, for an endfire array with 0.425λ spacing, the progressive phases in the neighboring elements differ by 0.425 × 360° or 153°.

will be given in Section IV when spacing-perturbation techniques are discussed.

Besides the linear array, the circular array represents another class of arrays with a simple geometry and important practical applications. Figure 2 shows a general circular array with nonuniformly spaced elements in the xy-plane. Substituting $\theta_n = \pi/2$ in (3), we obtain

$$\cos \alpha_n = \sin \theta \cos (\phi-\phi_n) \ . \tag{19}$$

Since $r_n = \rho_n = \rho$ for all n, b_{mn} in (10) simplifies to

$$b_{mn} = \frac{1}{4\pi} \int_0^{2\pi} d\phi \int_0^{\pi} g(\theta,\phi) \exp\{-jk\rho_{mn} \cos(\phi-\phi_{mn}) \sin \theta\} \sin \theta d\theta , \tag{20}$$

where

$$\rho_{mn} = 2\rho \left|\sin(\phi_m-\phi_n)/2\right| \tag{21}$$

and

$$\phi_{mn} = \tan^{-1} \left(\frac{\sin \phi_m - \sin \phi_n}{\cos \phi_m - \cos \phi_n}\right) . \tag{22}$$

For a circular array of N uniformly spaced isotropic elements, $g(\theta,\phi) = 1$, $\phi_n = 2n\pi/N$, and

$$\rho_{mn} = 2\rho \left|\sin (m-n)\pi/N\right| . \tag{23}$$

We have, from (20),

$$b_{mn} = \frac{\sin (k\rho_{mn})}{k\rho_{mn}} \tag{24}$$

The expressions of b_{mn} for some directive elements with typical power-pattern functions have also been given [13].

Figure 3 compares the maximum directivity D_M with the directivity D_u under a uniform-amplitude and cophasal excitation for a 12-element circular array as a function of the array diameter. We note that D_M is everywhere higher than D_u and that D_M increases very rapidly when the array diameter is less than 2λ (a superdirective situation). Since superdirective arrays require very large currents of opposite signs in neighboring elements, resulting in excessive heat loss and very low radiation intensity in the direction of the main beam, it is appropriate to define a <u>main-beam radiation efficiency</u> η. A practical optimum design then would be a suitable compromise between D_M and η. We define

$$\eta = \frac{|E(\theta_o, \phi_o)|^2}{N \sum_{n=1}^{N} I_n^2} \times 100\% . \tag{25}$$

By the use of Schwarz's inequality, it is easy to show that [13] η equals 100% only for uniformly excited cophasal arrays and becomes very small under superdirective situations. The main-beam radiation efficiency turns out to be the reciprocal of the tolerance sensitivity used by Uzsoky and Solymar [10] to measure the mean-square variation of the maximum field with respect to the mean-square deviation of the excitation. The values of η under the conditions for D_M are also plotted in Fig. 3.

It has been pointed out [16] that the optimization problem becomes easier by applying an orthogonalization process for arrays possessing a cyclic symmetry. For large circular arrays with many uniformly spaced elements, the tedium of inverting a large $\underset{\sim}{B}$ matrix (in (14)) can be circumvented through the introduction of a rotational operator [31].

IV. OPTIMIZATION BY SPACING ADJUSTMENTS

In the preceding section optimization was achieved by adjusting the excitation amplitudes and phases in the fixed elements of a given array. If the element positions are also allowed to vary, we acquire an additional dimension of freedom (which represents an additional N-1 degrees of freedom for a linear array with N elements), and we expect to be able to improve on the results obtained by excitation adjustments only. A spacing-perturbation technique exists which is useful for optimizing the directivity in a given direction or the signal-to-noise ratio in a given noise environment [6]. This technique will be developed in this section.

Consider a linear array of N identical elements symmetrically located about the origin along the x-axis. Let (θ_o, ϕ_o) be the direction of the main beam, and $I_n \exp(j\Phi_n)$ and $I_n \exp(-j\Phi_n)$ be the excitations in the nth and -nth elements respectively, where

$$\Phi_n = kd_n \sin\theta_o \cos\phi_o + \psi_n . \tag{26}$$

In (26), d_n is the distance of the nth element from the origin, and ψ_n is the phase shift from cophasal operation. Note that, except for the assumed symmetry about the origin, there is no restriction on element spacings which may be nonuniform and the elements themselves need not be omnidirectional. If N is odd* and equals 2M-1, the array factor is

*Only very minor modifications are needed when N is even. The formulation is entirely parallel.

$$E(u) = I_o + 2 \sum_{n=1}^{M} I_n \cos(\delta_n u + \psi_n) \tag{27}$$

where

$$u = kd(\sin\theta\cos\phi - \sin\theta_o\cos\phi_o) \tag{28}$$

$$\delta_n = d_n/d \tag{29}$$

and d, a normalizing distance, may be any choice of convenience. For example, if one starts with an equally spaced array, it would be natural to make d the uniform spacing between neighboring elements. We write the output signal-to-noise ratio in the direction of the signal ($\theta = \theta_o$, $\phi = \phi_o$, $u = 0$) as

$$SNR = \frac{|E(0)|^2}{\frac{1}{4\pi}\int_0^{2\pi} d\phi \int_0^{\pi} |E(u)|^2\, w(\theta,\phi)\sin\theta d\theta} \tag{30}$$

where $w(\theta,\phi)$ is a weighting function defined in (11). Our problem is to find the set of normalized element positions $\{\delta_n\}$ such that SNR is maximized for a given set of excitation amplitudes and phases. Note that this becomes a directivity maximization problem when $w(\theta,\phi) = g(\theta,\phi)$.

Let $\{\delta_n^o\}$ denote the original normalized element positions, and

$$\delta_n = \delta_n^o + x_n \tag{31}$$

where x_n represents the spacing perturbation for the nth element and $x_n \ll 1$. Substitution of (31) in (27) yields approximately

$$E(u) = E^o(u) - 2 \sum_{n=1}^{M} I_n x_n u \sin(\delta_n^o u + \psi_n) \tag{32}$$

where $E^o(u)$ is the original, unperturbed array factor with δ_n^o substituted for δ_n in (27). Using (32), we can write (30) in the following form:

$$\mathrm{SNR} = \frac{|E^o(0)|^2}{\alpha - 2\underset{\sim}{x}'\underset{\sim}{\beta} + \underset{\sim}{x}'\underset{\sim}{C}\,\underset{\sim}{x}} \tag{33}$$

where

$$\alpha = \frac{1}{4\pi}\int_0^{2\pi} d\phi \int_0^{\pi} |E^o(u)|^2\, w(\theta,\phi) \sin\theta d\theta \tag{34}$$

$$\underset{\sim}{x}' = [x_1, x_2, \ldots x_n, \ldots, x_N] \tag{35}$$

is the transpose of the column matrix of spacing perturbations $\underset{\sim}{x}$; $\underset{\sim}{\beta}$ is a column matrix of typical element

$$\beta_n = \frac{1}{2\pi}\int_0^{2\pi} d\phi \int_0^{\pi} I_n u E^o(u) w(\theta,\phi) \sin(\delta_n^o u + \psi_n) \sin\theta d\theta; \tag{36}$$

and $\underset{\sim}{C} = [C_{mn}]$ is an N×N square matrix with

$$C_{mn} = \frac{1}{\pi}\int_0^{2\pi} d\phi \int_0^{\pi} I_m I_n u^2\, w(\theta,\phi) \sin(\delta_m^o u + \psi_m) \sin(\delta_n^o u + \psi_n) \sin\theta d\theta \, . \tag{37}$$

It can readily be shown that $\underset{\sim}{C}$ is symmetric and positive definite. Use can then be made of the following theorem which is proved in the Appendix.

Theorem 2 - If a quantity SNR can be expressed in terms of an $N \times 1$ real column vector $\underset{\sim}{x}$ as in (33), where α is a constant, $\underset{\sim}{\beta}$ is another $N \times 1$ real column vector, and $\underset{\sim}{C}$ is an $N \times N$ positive definite, symmetric, square matrix, then

$$\underset{(\underset{\sim}{x}=\underset{\sim}{x}_M)}{\text{Max}} \quad \text{SNR} = \frac{|E^o(0)|^2}{\alpha - \underset{\sim}{\beta}\,\underset{\sim}{C}^{-1}\underset{\sim}{\beta}} : \tag{38}$$

and

$$\underset{\sim}{x}_M = \underset{\sim}{C}^{-1}\underset{\sim}{\beta}. \tag{39}$$

Equations (38) and (39) give the results of a first-order perturbation. After the components of $\underset{\sim}{x}_M$ have been determined from (39), one can then use $(\underset{\sim}{\delta}^o + \underset{\sim}{x}_M)$ as the new normalized element-position column matrix and perform a second-order perturbation to obtain further improvement in the performance index. This process can be repeated until it becomes evident that further iteration yields a negligible improvement. The final values of $\{\delta_n\}$ determine the element positions for a maximum SNR for the given excitation. Now this perturbed nonuniformly spaced array can be further optimized by proper amplifications and phase shifts in the array elements using the method developed in the preceding section [6]. A second local maximum will be reached, which may possibly be further improved by holding the excitation unchanged and again perturbing the spacings. The cycle may be repeated until further adjustments are no longer worthwhile.

We illustrate the application of the above technique with a broadside array of 7 isotropic elements. It is desired to optimize the array for maximum SNR in a noise environment, $T(u)$ specified by Fig. 4 with $a=15$,

$u_1 = (1/4)(2\pi d/\lambda)$ and $u_2 = (1/12)(2\pi d/\lambda)$. The normalizing distance, d, is chosen to be 0.885λ which corresponds to the spacing for maximum directivity in a uniformly spaced linear array with 7 isotropic elements. The results for the (a) space-perturbed, (b) excitation-adjusted (by amplification), and (c) optimized arrays are listed in Table 1. We note that large improvements in SNR are possible by optimization through either

Table 1. SNR Optimization for Seven-Element Broadside Array

$$u_1 = \frac{1}{4}\left(\frac{2\pi d}{\lambda}\right), \quad u_2 = \frac{1}{12}\left(\frac{2\pi d}{\lambda}\right), \quad d_0 = 0$$

	I_0	I_1	I_2	I_3	$(d_1-d_0)\frac{2}{\lambda}$	$(d_2-d_1)\frac{2}{\lambda}$	$(d_3-d_2)\frac{2}{\lambda}$	SNR
Uniform array	1.00	1.00	1.00	1.00	1.77	1.77	1.77	16.0
Space-perturbed array	same as above				1.65	1.76	2.10	19.8
Exc.-adjusted perturbed array	1.00	0.89	0.67	0.39	same as above			78.1
Optimized array	1.00	0.86	0.59	0.40	1.66	1.72	1.74	181.9

spacing perturbation or excitation adjustments, and that the SNR of the optimized array is about 11.4 times that of the original uniform array. Of course, the array optimized for a maximum SNR is not the same as the one for a maximum directivity. The directivity of the SNR-optimized array in Table 1 (sketched in Fig. 5(a)) is 10.6, whereas a directivity-optimized broadside array of 7 isotropic elements has a directivity of 10.63 [6].

The radiation patterns of the SNR-optimized array are plotted in Fig. 5(b), where the spatial distribution of the noise or interference power is also shown. It is interesting to see that, at the expense of a slightly wider main beam, the sidelobes of the optimized array are everywhere lower than those of the uniform array. In particular, the first sidelobe, which normally occurs in a region where the noise power is high, is much suppressed and its position slightly shifted.

V. OPTIMIZATION BY PHASE ADJUSTMENTS

In Section III we discussed the method for maximizing array directivity by adjusting the excitation amplitudes and phases in the array elements. Accurate amplitude adjustments require the use of precision amplifiers or attenuators, or specially designed directional couplers. It is hence of both theoretical and practical interest to develop a technique for optimizing the directivity of uniformly excited arrays requiring only phase adjustments. The elimination of the need for amplitude adjustments would result in a simplified feed structure and a reduced cost.

Directivity maximization under the constraint of a uniform amplitude in all the array elements can be formulated by Lagrange multiplier methods in several ways. However, it was found that the resulting equations were not amenable to a stable solution even by iterative methods, because of convergence problems. On the other hand, a perturbation procedure similar to that employed in Section IV for spacing adjustments can be used for the phase-adjustment problem [18]. The essential steps of this procedure will be developed in the following.

Consider a linear array of 2M+1 symmetrically located, equally spaced, identical elements, all with the same excitation amplitude I. The array factor is then, from (27),

$$E(u) = I[1 + 2 \sum_{n=1}^{M} \cos(nu + \psi_n)] , \tag{40}$$

where u is defined in (28) and ψ_n is the phase shift from cophasal operation. We may write

$$\psi_n = \psi_n^o + x_n \tag{41}$$

where ψ_n^o is an assumed initial value and $x_n << 1$ is a small perturbation. Ordinarily it is convenient to start from a cophasal excitation, i.e., $\psi_n^o = 0$. Substituting (41) in (40), we obtain approximately

$$E(u) = E^o(u) - 2I \sum_{n=1}^{M} x_n \sin(nu + \psi_n^o) \ . \tag{42}$$

Using (42) in (4), we can express the directivity of the array in the following form:

$$D = \frac{\frac{1}{I^2} |E(\theta_o, \phi_o)|^2}{\alpha_1 - \underset{\sim}{x}'\underset{\sim}{\beta}_1 + \underset{\sim}{x}'\underset{\sim}{C}_1 \underset{\sim}{x}} \tag{43}$$

where

$$\underset{\sim}{x}' = [x_{-M}, \ldots, x_{-1}, x_0, x_1, \ldots, x_M] \tag{44}$$

and α_1, $\underset{\sim}{\beta}_1$, and $\underset{\sim}{C}_1$ are similar to α, $\underset{\sim}{\beta}$, and $\underset{\sim}{C}$ in (33). α_1 is identical to α in (34), and the elements of $\underset{\sim}{\beta}_1$ and $\underset{\sim}{C}_1$ can be obtained from (36) and (37) respectively with $I_m = I_n = I$ and the argument $(\delta_n^o u + \psi_n)$ of the sine functions replaced by $(nu + \psi_n^o)$. Element power-pattern function $g(\theta,\phi)$ replaces weighting function $w(\theta,\phi)$ in computing directivity.

It is now obvious that the same Theorem 2, which is proved in the Appendix and found useful for optimization by spacing adjustments, can be used for maximizing D in (43) by phase adjustments. The required phase changes in the array elements are determined from

$$\underset{\sim}{x}^1 = \underset{\sim}{C}_1^{-1} \underset{\sim}{\beta}_1 \tag{45}$$

which represents a first-order perturbation from the initial values $\{\psi_n^o\}$. One may then treat $\{\psi_n^o + x_n^1\}$ as the new initial values and perform

a second-order perturbation. This process may be repeated until it is apparent that a maximum directivity has been obtained.

Although the preceding formulation for optimization by phase adjustments starts with a linear array, it is clear that the procedure can be applied to an arbitrary three-dimensional array. In particular, for a circular array of N elements with radius ρ in the xy-plane, a phase perturbation as indicated in (41) results in an array factor

$$E(\theta,\phi) = E^{o}(\theta,\phi) - 2I \sum_{n=1}^{N} x_n \sin(\Delta_n + \psi_n^{o}) \qquad (46)$$

with

$$\Delta_n = k\rho[\sin\theta\cos(\phi-\phi_n) - \sin\theta_o\cos(\phi_o-\phi_n)] \qquad (47)$$

where ϕ_n denotes the location of the nth element and (θ_o,ϕ_o) is the direction of the main beam. We note that (46) is entirely similar to (42). Substitution of (46) in (4) will yield a directivity expression in the form of (43), and hence the same optimization procedure follows. The optimum directivity, D_o, obtained by phase adjustments only for a circular array with 12 uniformly spaced short dipoles is plotted in Fig. 6 as a function of array diameter. In the same figure are also plotted D_M the maximum directivity when both amplitudes and phases are adjusted, and D_u, the directivity under a uniform-amplitude and cophasal excitation. We note that the D_o curve lies everywhere between the D_M and D_u curves. For most array diameters less than 3λ (element spacing less than $3\lambda/4$) an improvement of about 2 dB in directivity is possible by phase adjustments alone. When the array diameter is very small, the directivity of the phase-adjusted array increases rapidly, indicating a superdirective situation which is absent in a uniform cophasal array. The main-beam radiation efficiency of a superdirective array tends to be very low, as has been pointed out in Section III.

VI. OPTIMIZATION WITH CONSTRAINTS

In previous sections we discussed techniques for maximizing the directivity or the signal-to-noise ratio of an antenna array without constraints; that is, without imposing at the same time a requirement on any other performance index of the array. We have already seen that a maximum directivity for an array with closely spaced elements is accompanied by a low main-beam radiation efficiency. In practice, we may desire that a maximum directivity be obtained together with a prescribed value for main-beam radiation efficiency. The existence of constraints or auxiliary conditions effectively reduces the total number of independent variables which can be adjusted for optimization. In such cases, a procedure using Lagrange multipliers can be applied to determine the stationary value of directivity [19]. This approach has been employed [5] to maximize signal-to-noise ratio under a constraint on a Q-factor. The Q-factor is a quantity which is proportional to the ratio of the directivity and the main-beam radiation efficiency. It turns out that this approach results in a rather involved numerical procedure. We shall not go into it further here.

A more useful class of optimization problems with constraints pertains to the maximization of some performance index while controlling the array pattern in certain definite ways. For instance, one may wish to have a maximum directivity in the direction of some distant transmitter or receiver and, at the same time, to minimize the interference from some other directions. This is equivalent to the problem of directivity maximization with controlled locations of certain pattern nulls, and can be reduced to that of maximizing

the ratio of two Hermitian forms, as studied in Section III, through the introduction of a constraint matrix.

The array factor in (1) can be written as the inner product of the space vector $\underset{\sim}{F} = [\exp(-jkr_n \cos \alpha_n)]$ and the excitation vector $\underset{\sim}{J}$ defined in (5); i.e.,

$$E(\theta,\phi) = \langle \underset{\sim}{F}, \underset{\sim}{J} \rangle = \underset{\sim}{F}^{\dagger} \underset{\sim}{J} \quad . \tag{48}$$

Pattern nulls in the directions (θ_i, ϕ_i) are specified by homogeneous equations

$$\underset{\sim}{F}_i^{\dagger} \underset{\sim}{J} = \sum_{n=1}^{N} I_n \exp[j(\psi_n + kr_n \cos \alpha_{in})] = 0 \ , \tag{49}$$

$$i = 1, 2, \ldots, M < (N-1)$$

where α_{in} are obtained from (2) by setting $\theta = \theta_i$ and $\phi = \phi_i$. A geometrical interpretation of (49) is that the excitation vector $\underset{\sim}{J}$ which we seek to maximize the directivity in the direction (θ_o, ϕ_o) is now required to be simultaneously orthogonal to M independent constraint vectors $\underset{\sim}{F}_i$. The N-dimensional space is divided into two mutually orthogonal subspaces: an M-dimensional subspace containing the constraint vectors and an (N-M)-dimensional subspace where the excitation vector $\underset{\sim}{J}$ must lie. The mathematical procedure [21] for maximization under constraints consists of (a) finding an appropriate set of M mutually orthogonal vectors that occupy the same subspace as the constraint vectors $\underset{\sim}{F}_1, \underset{\sim}{F}_2, \ldots, \underset{\sim}{F}_M$, (b) obtaining an additional orthogonal (N-M) vectors which at the same time are orthogonal to the first M vectors, (c) forming

from steps (a) and (b) an N by N normalized constraint matrix which is the unitary matrix for coordinate transformation, (d) transforming the directivity expression as a ratio of two Hermitian forms to the new orthogonal coordinate system, and (e) maximizing the directivity in the same manner as outlined in Section III.

The Gram-Schmidt procedure [19] provides a method for determining an orthonormal basis for a vector space in which any set of spanning vectors is known. This method can be used to find the constraint matrix in steps (a) and (b). Assuming $\underset{\sim}{P}$ to be the normalized constraint matrix which is also the transformation matrix, we write

$$\underset{\sim}{P}\,\underset{\sim}{J} = \underset{\sim}{J}_c \; . \tag{50}$$

The directivity in (7) becomes, after the coordinate transformation,

$$D = \frac{\underset{\sim}{J}_c^{\dagger}\,(\underset{\sim}{P}\,\underset{\sim}{A}\,\underset{\sim}{P}^{\dagger})\underset{\sim}{J}_c}{\underset{\sim}{J}_c^{\dagger}\,(\underset{\sim}{P}\,\underset{\sim}{B}\,\underset{\sim}{P}^{\dagger})\underset{\sim}{J}_c} = \frac{\underset{\sim}{J}_c^{\dagger}\,\underset{\sim}{A}_c\,\underset{\sim}{J}_c}{\underset{\sim}{J}_c^{\dagger}\,\underset{\sim}{B}_c\,\underset{\sim}{J}_c} \; . \tag{51}$$

Inasmuch as each of the first M mutually orthogonal vectors is a linear combination of the constraint vectors, the first M rows of $\underset{\sim}{J}_c$ in (50) are linear combinations of the homogeneous constraint equations (49) and are therefore zeros. Hence the first M entries in $\underset{\sim}{J}_c$ and the first M rows and M columns of $\underset{\sim}{A}_c$ and $\underset{\sim}{B}_c$ can be discarded, resulting in an abridged form for directivity [20]:

$$D_a = \frac{\underset{\sim}{J}_a^{\dagger}\,\underset{\sim}{A}_a\,\underset{\sim}{J}_a}{\underset{\sim}{J}_a^{\dagger}\,\underset{\sim}{B}_a\,\underset{\sim}{J}_a} \quad , \tag{52}$$

where $\underset{\sim}{A}_a$ and $\underset{\sim}{B}_a$ are the abridged (N-M) by (N-M) matrices and $\underset{\sim}{J}_a$ is the

abridged (N-M)-element column vector. The remainder of the optimization process then follows in exactly the same manner as that pertaining to the unconstrained D in (7), Section III, except that the numerical problem is now simpler because the matrices involved are of a lower rank It is obvious that the maximum obtainable directivity with constraints will be less than that without constraints on account of the reduced freedom. Typical results on maximum directivity with null placements and with reduced radiation level in an angular sector have been published [20].

VII. CONSIDERATION OF MUTUAL COUPLING

One tacit assumption implied in the optimization techniques which we have considered thus far is that all the elements in an array have the same radiation pattern. This is tantamount to assuming that mutual-coupling effects are negligible. For large arrays with many elements this assumption is acceptable if our interest lies in array directivity and not in the current distribution or the exact radiation pattern of each element. The consideration of mutual-coupling effects in array optimization greatly complicates the problem. To the author's knowledge, no work has been published on array synthesis or optimization for mutually coupled aperture-type radiators. With wire antennas the method of moments can be used to obtain numerical answers [23-26]. In this section we will outline the optimization procedure for arrays of wire antennas when mutual coupling is not neglected.

The moment method for solving electromagnetic problems consists mainly of three steps; namely, the formulation of the governing integro-differential equations, the expansion of the unknown functions in terms of a set of linearly independent basis functions, and the testing of the expanded equation by forming inner products with a set of linearly independent set of weighting functions [26]. The result is a set of simultaneous equations which can be solved by matrix methods. The choice of the basis and weighting functions depends on the desired accuracy and the ease in evaluating the coefficients in the simultaneous equations. One convenient technique in the choice of basis functions is to divide the domain of the unknown function into small

intervals or subsections. Simple basis functions such as pulses or triangles are defined to exist over one or a few such subsections and to be zero elsewhere. The simplest weighting functions are Dirac delta functions defined at discrete points at which the expanded governing equations are to be satisfied. This is the point-matching or sampling method.

For thin wire antennas, the currents and charges on the wires can be approximated by current and charge filaments along the antenna axes. We consider an array of thin linear antennas parallel to the z-axis. From the Maxwell's equation for time-harmonic fields in a homogeneous medium

$$\bar{\nabla} \times \bar{H} = j\omega\varepsilon \, \bar{E} + \bar{J} \tag{53}$$

and $\bar{H} = \frac{1}{\mu} \bar{\nabla} \times \bar{A}$, we have

$$\bar{E} = \frac{1}{j\omega\varepsilon} \left(\frac{1}{\mu} \bar{\nabla} \times \bar{\nabla} \times \bar{A} - \bar{J}\right). \tag{54}$$

Now

$$\bar{\nabla} \times \bar{\nabla} \times \bar{A} = \bar{\nabla} \, \bar{\nabla} \cdot \bar{A} - \nabla^2 \bar{A} \tag{55}$$

and

$$(\nabla^2 + k^2) \, \bar{A} = -\mu \bar{J}, \tag{56}$$

where $k^2 = \omega^2 \mu\varepsilon$. Combination of (55) and (56) with (54) yields

$$\bar{E} = \frac{1}{j\omega\mu\varepsilon} (\bar{\nabla} \, \bar{\nabla} \cdot \bar{A} + k^2 \bar{A}) \; . \tag{57}$$

Since $\bar{J}$ has only a z-component, we can rewrite (57) as a scalar equation

$$E_z = \frac{1}{j\omega\mu\varepsilon} \left(\frac{\partial^2}{\partial z^2} + k^2\right) A_z \; . \tag{58}$$

For thin linear antennas, the tangential electric field E_{zp} at the center of a typical pth subsection is then [25]

$$E_{zp} = \frac{1}{j\omega\varepsilon}\left(\frac{\partial^2}{\partial z^2} + k^2\right) \int\limits_{\substack{\text{all}\\ \text{antennas}}} \frac{I(z')\exp[-jk|\bar{r}_p - \bar{r}'|]}{4\pi|\bar{r}_p - \bar{r}'|} dz' , \tag{59}$$

where $\bar{r}_p$ is the position vector to the center of the pth subsection under consideration, and $\bar{r}'$ is a vector from the origin to a point on an antenna at which the current is $I(z')$. If each of the N antennas in the array is divided into S subsections of length $\Delta\ell_q$ which carries a constant current I_q, (59) can be approximated by a summation:

$$E_{zp} = \sum_{q=1}^{M} I_q \left\{ \frac{1}{j\omega\varepsilon}\left(\frac{\partial^2}{\partial z^2} + k^2\right) \int\limits_{\Delta\ell_q} \frac{\exp[-jk|\bar{r}_p - r'|]}{4\pi|\bar{r} - \bar{r}'|} dz' \right\} . \tag{60}$$

In (60), $M = NS$. The quantity in the wavy brackets is the electric field at the center of subsection p due to a unit current in subsection q and can be written as $Z_{pq}/\Delta\ell$, where equal subsections ($\Delta\ell_q = \Delta\ell$) are assumed for simplicity. Thus,

$$E_{zp}(\Delta\ell) = V_p = \sum_{q=1}^{M} Z_{pq} I_q , \tag{61}$$

or, in matrix form,

$$\underset{\sim}{V} = \underset{\approx}{Z}\, \underset{\sim}{I} , \tag{62}$$

where $\underset{\sim}{V} = [V_p]$ and $\underset{\sim}{I} = [I_q]$ are M by 1 column matrices and $\underset{\sim}{Z} = [Z_{pq}]$ is an M by M square matrix. $\underset{\sim}{Z}$ may be called a generalized impedance matrix and depends only on the geometrical configuration of the array.

In the terminology of the method of moments, (62) is the result of (a) using pulse basis functions each of which exists over only one sub-

section, (b) using an integral-type inner product, and (c) using Dirac delta functions as weighting functions. The differential operator in (60) may be approximated by a second-order difference operator and computer subroutines for the calculation of Z_{pq} are available [25]. In a radiation problem $\underset{\sim}{V}$ is a column matrix of known voltages which are all zero except for the excitation voltages at feed points. Column matrix $\underset{\sim}{I}$ determines the current distributions on the wire antennas:

$$\underset{\sim}{I} = \underset{\sim}{Z}^{-1}\underset{\sim}{V} = \underset{\sim}{Y}\,\underset{\sim}{V} \; . \tag{63}$$

With $\underset{\sim}{I}$ known, all field quantities of interest can be determined.

The array directivity defined in (4) can be expressed in terms of the total power input to the array, P_{in}, if the antenna wires are assumed to be perfectly conducting. The electric field in the far zone of an array consisting of z-directed wire antennas is

$$E_\theta = - j\omega A_\theta = j\omega A_z \sin\theta \tag{64}$$

Thus,

$$\begin{aligned} D &= \frac{4\pi r^2 |E(\theta_o)|^2/2}{\zeta P_{in}} \\ &= \frac{2\pi r^2 |\omega A_z \sin\theta_o|^2}{\zeta P_{in}} \, , \end{aligned} \tag{65}$$

where ζ is the intrinsic impedance of the medium. The time-average input power to the array is

$$P_{in} = \frac{1}{2} \operatorname{Re} \sum (V_{in} I_{in}^*). \tag{66}$$

Since nonzero voltages exist only at the feed points, (66) can be written as

$$P_{in} = \frac{1}{2} \mathrm{Re}\, (\underset{\sim}{I}^{+}\underset{\sim}{V})$$

$$= \frac{1}{2} \mathrm{Re}\, (\underset{\sim}{V}^{+}\underset{\sim}{Y}^{+}\underset{\sim}{V})$$

$$= \frac{1}{2}\, \underset{\sim}{V}_1^{\dagger} \left[\frac{\underset{\sim}{Y}_1 + \underset{\sim}{Y}_1^{*}}{2}\right] \underset{\sim}{V}_1 \tag{67}$$

where $\underset{\sim}{V}_1$ is a column matrix reduced from $\underset{\sim}{V}$ by retaining only the N (number of wire antennas in the array) nonzero feed-point voltages, and $\underset{\sim}{Y}_1$ is an N×N admittance matrix reduced from $\underset{\sim}{Y}$ by deleting all elements that do not correspond to the feed subsections.

In the far zone, the magnetic potential can be approximated by

$$A_z = \frac{\mu(\Delta\ell)}{4\pi r} \exp(-jkr) \sum_{q=1}^{M} I_q \exp[jkr_q \cos \alpha_q] \tag{68}$$

where α_q is the angle between the position vector to the qth subsection carrying current I_q and that to the field point. Using the matrix representations of $\underset{\sim}{F}$ as in (6) and of $\underset{\sim}{I}$ as in (62), we can write (68) as

$$A_z = \frac{\mu(\Delta\ell)}{4\pi r} \exp(-jkr)\, \underset{\sim}{F}\, \underset{\sim}{I} \tag{69}$$

or in reduced matrices as

$$A_z = \frac{\mu(\Delta\ell)}{4\pi r} \exp(-jkr)\, \underset{\sim}{F}_1^{\dagger}\, \underset{\sim}{Y}_1 \underset{\sim}{V}_1 \,. \tag{70}$$

For simplicity we write

$$\underset{\sim}{G}_1^{\dagger} = \underset{\sim}{F}_1^{\dagger}\, \underset{\sim}{Y}_1 \,. \tag{71}$$

With (67), (70) and (71), D in (65) becomes

$$D = \frac{(\omega\mu\Delta\ell \sin \theta_o)^2}{4\pi\zeta} \frac{\underset{\sim}{V}_1^{\dagger}\, \underset{\sim}{G}_1\, \underset{\sim}{G}_1^{\dagger}\, \underset{\sim}{V}_1}{\underset{\sim}{V}_1^{\dagger} \left[\frac{\underset{\sim}{Y}_1 + \underset{\sim}{Y}_1^{*}}{2}\right] \underset{\sim}{V}_1} \,. \tag{72}$$

It can be shown [23] that the excitation voltage matrix required for making D in (72) a maximum is

$$\underset{\sim}{V}_1 = \left[\frac{\underset{\sim}{Y}_1 + \underset{\sim}{Y}_1^*}{2}\right]^{-1} \underset{\sim}{G}_1 \tag{73}$$

and

$$D_M = \frac{(\omega\mu\Delta\ell)^2}{4\pi\eta} \underset{\sim}{G}_1^{\dagger} \left[\frac{\underset{\sim}{Y}_1 + \underset{\sim}{Y}_1^*}{2}\right] \underset{\sim}{G}_1 . \tag{74}$$

The directivity optimization problem with voltage excitation is now completely solved, and the effect of mutual coupling has been taken into consideration. As can be seen, the main task in obtaining numerical solutions lies in the determination of the admittance matrix $\underset{\sim}{Y}_1$.

Using the above procedure, Cummins [23] determined the maximum directivity in the principal H-plane of a circular array of 4 uniformly spaced center-fed wire antennas. Figure 7 shows the variation of maximum directivity (D_M) versus antenna length ($2h/\lambda$) with array diameter (d/λ) as the parameter. The points marked by crosses correspond to the maximum directivity of a circular array of 4 isotropic sources as computed by the method outlined in Section III.

The solution of the dual problem of an array excited by a set of current sources follows an entirely similar procedure [23].

VIII. OTHER CONSIDERATIONS

We discuss here several related aspects of the array optimization problem which either can be treated by an extension of some of the preceding techniques or need special attention.

(a) Maximization of Power Gain - When the antenna wires are not perfectly conducting, resistive losses occur and power gain is no longer the same as directivity or directive gain. Electromagnetically speaking, E_{zp} in (61) on the surface of subsection p is no longer zero, but is equal to the product of I_p and Z_i, the internal impedance per unit length of the wire conductor.

$$V_p = E_{zp}(\Delta\ell) = Z'_{pp} I_p \ , \tag{75}$$

where

$$Z'_{pp} = Z_i(\Delta\ell) \tag{76}$$

and

$$Z_i = \left(\frac{j\omega\mu}{\sigma}\right)^{1/2} = \frac{1+j}{\sigma\delta} \ . \tag{77}$$

In (77), σ and δ are respectively the conductivity and the skin depth of the wire conductor at the operating frequency. Hence, if the generalized impedance matrix $\underset{\sim}{Z}$ for perfectly conducting wires has been found, the only modification needed for a moment solution with finitely conducting wires is the addition of $Z'_{pp} = Z_i(\Delta\ell)$ to the diagonal elements of $\underset{\sim}{Z}$.

It is obvious that the achievable maximum power gain for a given array configuration is lower than the maximum directivity because of the resistive losses. Numerical results have also shown [23] that for arrays with closely

spaced elements the required excitations for maximum power gain are not the same as those for maximum directivity.

(b) Arrays with Random Errors - Under practical applications random errors exist in excitation amplitudes and phases as well as in element positions. It is then of interest to examine the effect of random errors in these design parameters on the optimization procedure. This has been done for arrays with an arbitrary geometrical configuration [27-30]. Correlations are allowed to exist between the errors in the array parameters and no restrictions are necessary either on the magnitude or on the probability distribution of the random errors. The dependence of the expected directivity or SNR, the main-beam radiation efficiency, the optimum excitation amplitudes and phases and the radiation pattern on the variance and correlation distance of parameter errors has been studied [29,30]. It was found that the excitations calculated on the basis of no random errors do not yield a maximum expected directivity when parameter errors exist.

(c) Techniques for Large Arrays - In the numerical solution of the array optimization problem by the method of moments in Section VII, it is necessary to invert the generalized impedance matrix $\underset{\sim}{Z}$, as defined in (62). The order of the matrix $\underset{\sim}{Z}$ is M=NS, which is the product of the number of wire antennas in the array and the number of subsections for each antenna (assuming equal antenna lengths). Hence M can be very large for large arrays with many elements, especially when the elements are of a length which is an appreciable fraction of the operating wavelength. The feasibility of inverting large matrices is constrained by computer memory

capacity and the cost. Alternative techniques which will relax these constraints are therefore of great importance.

For circular arrays with many uniformly spaced elements, the inversion of the relevant large matrix is facilitated through diagonalization by a change of basis with a unitary transformation matrix [31]. The columns of the transformation matrix are the eigenvectors of a rotational operator. With this technique the inversion of the large matrix can be evaluated by straightforward matrix multiplication.

For array configurations with no rotational symmetry other methods must be sought. The far-zone electric field due to an array of N parallel z-directed dipoles is

$$E(\theta,\phi) = \frac{j\omega\mu}{4\pi r} e^{-j\beta r} \sum_{n=1}^{N} e^{j\beta \bar{r}_n \cdot \hat{u}} \int_{-h}^{h} I_n(z') e^{j\beta z' \cos\theta} \sin\theta \, dz' \quad , (78)$$

where β is the phase constant, $\bar{r}_n$ is the vector from the origin to the center of the nth dipole, and $\hat{u}$ is the unit vector from the origin to the observation point. The first step in the optimization problem is the insertion of (78) in the expression for the performance index of interest. In order to maximize the performance index, it is necessary to use an appropriate expansion for the currents $I_n(z')$. The moment method discussed in Section VII used expansion functions defined over subsections, which resulted in M by M matrices. An alternative is to use the three-term theory for cylindrical antennas developed by King and his associates [32]. This

has been done, and it has been found that, by using some special properties of the integrals involved, accurate numerical solutions of array optimization problems can be obtained by working with matrices of order N (not 3N as first suspected, where $N = M/S$). Details of this technique together with numerical results will be reported in Part (B).

APPENDIX to PART (A) - PROOF OF THEOREM 2

Let

$$P = \alpha - 2x'\beta + x'C\,x \;. \tag{A-1}$$

which is the denominator of the quantity in (32). It is necessary to prove that if C is positive definite P will be minimum at

$$x_M = C^{-1}\beta \tag{A-2}$$

and

$$\min_{(x=x_M)} P = \alpha - \beta'C^{-1}\beta \;. \tag{A-3}$$

Proof: If C is positive definite, it is known that [15]

$$(\beta'C^{-1}\beta)(x'Cx) \geq (x'\,\beta)^2 \tag{A-4}$$

or

$$x'Cx \geq \frac{1}{\beta'C^{-1}\beta}\,(x'\,\beta)^2 \;, \tag{A-5}$$

where the equality sign applies when

$$x = x_M = C^{-1}\beta \;. \tag{A-2}$$

Let $c = \beta'C^{-1}\,\beta > 0$, and $b = x'\,\beta$. We have, from (A-1) and (A-5)

$$P = A - 2b + x'Cx \geq A - 2b + \frac{b^2}{c} \;. \tag{A-6}$$

But

$$A - 2b + \frac{b^2}{c} = A - c + \frac{1}{c}\,(c-b)^2 \geq A - c \;. \tag{A-7}$$

Combining (A-6) and (A-7), we obtain

$$P \geq A - \beta'C^{-1}\beta \;, \tag{A-8}$$

where the equality sign holds with (A-2); hence theorem 2 is proved.

PART (B). INTEGRAL-EQUATION APPROACH FOR OPTIMIZING ARRAYS WITH MUTUAL COUPLING

I. INTRODUCTION

In Part (A) we have discussed various techniques for optimizing some chosen performance indices of antenna arrays. When the array elements are parallel wire antennas, the method of moments can be used for optimization which includes the effect of mutual coupling. As explained in Part (A) - Section VII, it was found convenient to incorporate the method of subsections to convert the governing integro-differential equations into matrix equations by using pulse expansion functions and impulsive weighting functions. For an N-element array each subdivided into S segments, an M by M ($M=N\times S$) generalized impedance matrix results which must be inverted in the optimization procedure. This inversion process presents practical difficulties when N and S are large, because of limitations in computer memory capacity and in allowable cost. In this Part (B), we present a new approach for array optimization with the consideration of mutual coupling that requires the inversion of only an N by N matrix. Since computer time (cost) required is proportional to the cubic power of the rank of the matrix, we achieve a saving by a factor of S^3. If S equals 10, for example, this means a 1000-fold saving, a considerable factor indeed.

The approach we take starts with an integral-equation formulation for the array in terms of the unknown current distributions on the array elements.

Instead of using the methods of moments and subsections, we expand the current distribution functions as superpositions of suitable sinusoidal functions. In particular, we make use the three-term theory developed by King and his associates [32]. The subsequent theoretical development is quite involved, but we have succeeded in a considerable reduction in the order (and rank) of the matrices involved. Numerical solutions for typical arrays can be obtained with a few dollars' worth of computer time.

II. INTEGRAL-EQUATION FORMULATION

We consider an array of N parallel, z-directed, center-fed dipole antennas each of radius a and half-length h. The typical nth dipole is center-driven by a delta-function generator of strength V_n. The dipole conductor is assumed to be perfectly conducting and $\beta a \ll 1$, where β is the phase constant. The integral equation for the nth dipole in terms of the currents in all the array elements is [32]

$$\sum_{m=1}^{N} \int_{-h}^{h} I_m(z')K_{mn}(z,z')dz' = -\frac{j}{30} [C_n \cos \beta z + \frac{V_n}{2} \sin \beta|z|] \tag{79}$$

where

$$K_{mn}(z,z') = \frac{e^{-j\beta r_{mn}}}{r_{mn}} \tag{80}$$

$$r_{mn} = [(z-z')^2 + b_{mn}^2]^{1/2} \tag{81}$$

$$b_{mn} = \begin{cases} a , & m=n \\ d_{mn} , & m\neq n. \end{cases} \tag{82}$$

Letting z=h in (79), we have

$$\sum_{m=1}^{N} \int_{-h}^{h} I_m(z')K_{mn}(h,z')dz' = -\frac{j}{30} [C_n \cos \beta h + \frac{V_n}{2} \sin \beta h]. \tag{83}$$

Eliminating $C_n \cos \beta h$ from (79) and (83), we get

$$\sum_{m=1}^{N} \int_{-h}^{h} I_m(z')[K_{mn}(z,z') \cos \beta h - K_{mn}(h,z') \cos \beta z]dz'$$

$$= \frac{j}{60} V_n \sin \beta (h - |z|). \tag{84}$$

Equation (84) can be rewritten as

$$\sum_{m=1}^{N} \int_{-h}^{h} I_m(z')K'_{mn}(z,z') \cos \beta h \, dz'$$

$$= \frac{j}{60} [V_n \sin \beta (h-|z|) + U_n(\cos \beta z - \cos \beta h)] , \qquad (85)$$

where

$$K'_{mn}(z,z') = K_{mn}(z,z') - K_{mn}(h,z') \qquad (86)$$

and

$$U_n = -j60 \sum_{m=1}^{N} \int_{-h}^{h} I_m(z')K_{mn}(h,z')dz' . \qquad (87)$$

With n=1,2,...,N, we obtain N simultaneous integral equations from (85).

III. THE THREE-TERM THEORY

King's three-term theory approximates the currents $I_m(z)$ with three parts: one part is a sinusoid maintained directly by the driving voltage, and the other two parts are a shifted cosine and a shifted cosine with half-angle arguments which are induced by coupling between different parts of the antennas.

$$I_m(z) = \sum_{k=1}^{3} A_m^{(k)} S_k(z) \tag{88}$$

where

$$S_1(z) = \sin \beta (h - |z|) \tag{89}$$

$$S_2(z) = \cos \beta z - \cos \beta h \tag{90}$$

$$S_3(z) = \cos \frac{1}{2} \beta z - \cos \frac{1}{2} \beta h \tag{91}$$

The integral on the left side of (85) possesses the following approximate properties for the currents in (88) - (91) for different ranges of βb_{mn} values [33]:

(a) For $\beta b_{mn} < 1$,

$$\int_{-h}^{h} I_m(z')K'_{mnR}(z,z') \cos \beta h \, dz' \cong \psi_1' \, I_m(z). \tag{92}$$

where the function $K'_{mnR}(z,z')$ denotes the real part of $K'_{mn}(z,z')$.

(b) For $\beta b_{mn} \geq 1$,

$$\int_{-h}^{h} S_1(z')K'_{mnR}(z,z') \cos \beta h \, dz' \cong \psi_2' \, S_2(z). \tag{93}$$

(c) For all values of βb_{mn},

$$\int_{-h}^{h} S_2(z')K'_{mnR}(z,z') \cos \beta h \, dz' \cong \psi_3' S_2(z) \tag{94}$$

$$\int_{-h}^{h} S_3(z')K'_{mnR}(z,z') \cos \beta h \, dz' \cong \psi_4' S_3(z) \tag{95}$$

$$\int_{-h}^{h} I_m(z')K'_{mnI}(z,z') \cos \beta h \, dz' \cong \psi_5' S_3(z), \tag{96}$$

where the function $K'_{mnI}(z,z')$ denotes the imaginary part of $K'_{mn}(z,z')$. The proportionality constants ψ_1', ψ_2', ψ_3', ψ_4' and ψ_5' in approximations (92) - (96) are best determined where the distribution functions in the integrands are at their maximum values.

Under normal circumstances $\beta b_{mn} > 1$ for $m \neq n$ and $\beta b_{nn} = \beta a \ll 1$. Substituting (88) - (91) in the left side of (85) and using (92) - (96), we have

(a) $m=n$, $b_{nn} = \beta a < 1$.

$$\int_{-h}^{h} A_n^{(1)} S_1(z')K'_{nn}(z,z') \cos \beta h \, dz'$$

$$= A_n^{(1)} \psi'^{(1)}_{nnR} S_1(z) + A_n^{(1)} \psi'^{(1)}_{nnI} S_3(z) \tag{97}$$

$$\int_{-h}^{h} A_n^{(2)} S_2(z') K'_{nn}(z,z') \cos \beta h \, dz'$$

$$= A_n^{(2)} \psi'^{(3)}_{nnR} S_2(z) + A_n^{(2)} \psi'^{(2)}_{nnI} S_3(z) \tag{98}$$

$$\int_{-h}^{h} A_n^{(3)} S_3(z')K'_{nn}(z,z') \cos \beta h \, dz'$$

$$= A_n^{(3)} \psi'^{(3)}_{nn} S_3(z). \tag{99}$$

(b) $m \neq n$, $\beta b_{mn} = \beta d_{mn} \geq 1$.

$$\int_{-h}^{h} A_m^{(1)} S_1(z')K'_{mn}(z,z') \cos \beta h \, dz'$$

$$= A_m^{(1)} [\psi'^{(1)}_{mnR} S_1(z) + \psi'^{(2)}_{mnR} S_2(z) + \psi'^{(1)}_{mnI} S_3(z)] \tag{100}$$

$$\int_{-h}^{h} A_m^{(2)} S_2(z')K'_{mn}(z,z') \cos \beta h \, dz'$$

$$= A_m^{(2)} [\psi'^{(3)}_{mnR} S_2(z) + \psi'^{(2)}_{mnI} S_3(z)] \tag{101}$$

$$\int_{-h}^{h} A_m^{(3)} S_3(z')K'_{mn}(z,z') \cos \beta h \, dz' = A_m^{(3)} \psi'^{(3)}_{mn} S_3(z). \tag{102}$$

The proportionality constants in (97) - (102) are:

$$\psi'^{(1)}_{mnR} = \begin{cases} \dfrac{\cos \beta h}{\sin \beta h} \displaystyle\int_{-h}^{h} S_1(z')[\frac{\cos \beta R_1}{R_1} - \frac{\cos \beta R_2}{R_2}]dz' , & \beta h < \pi/2 \\ \cos \beta h \displaystyle\int_{-h}^{h} S_1(z')[\frac{\cos \beta R_3}{R_3} - \frac{\cos \beta R_2}{R_2}]dz' , & \beta h > \pi/2 \end{cases} \tag{103}$$

with

$$R_1 = (z'^2 + b_{mn}^2)^{1/2} \tag{104}$$

$$R_2 = [(h-z')^2 + b_{mn}^2]^{1/2} \tag{105}$$

$$R_3 = [(h-z'-\lambda/4)^2 + b_{mn}^2]^{1/2} \tag{106}$$

$$\psi'^{(2)}_{mnR} = \frac{\cos \beta h}{1-\cos \beta h} \int_{-h}^{h} S_1(z')[\frac{\cos \beta R_1}{R_1} - \frac{\cos \beta R_2}{R_2}]dz' \tag{107}$$

$$\psi'^{(3)}_{mnR} = \frac{\cos \beta h}{1-\cos \beta h} \int_{-h}^{h} S_2(z')[\frac{\cos \beta R_1}{R_1} - \frac{\cos \beta R_2}{R_2}]dz' \tag{108}$$

$$\psi'^{(1)}_{mnI} = \frac{-\cos \beta h}{1-\cos \beta h/2} \int_{-h}^{h} S_1(z')[\frac{\sin \beta R_1}{R_1} - \frac{\sin \beta R_2}{R_2}]dz' \tag{109}$$

$$\psi'^{(2)}_{mnI} = \frac{-\cos \beta h}{1-\cos \beta h/2} \int_{-h}^{h} S_2(z')[\frac{\sin \beta R_1}{R_1} - \frac{\sin \beta R_2}{R_2}]dz' \tag{110}$$

$$\psi'^{(3)}_{mn} = \frac{\cos \beta h}{1-\cos \beta h/2} \int_{-h}^{h} S_3(z')[\frac{e^{-j\beta R_1}}{R_1} - \frac{e^{-j\beta R_2}}{R_2}]dz'. \tag{111}$$

When $\beta h = \pi/2$, expressions (109) - (111) become indeterminate and component current functions different from those given in (89) - (91) must be chosen. This special case will be discussed in Section V.

We are now ready to substitute the above in (85) and equate the coefficients of $S_k(z)$, $k=1,2,3$. We obtain 3 sets of equations ($n=1,2,\ldots,N$):

$$\sum_{m=1}^{N} A_m^{(1)} \psi'^{(1)}_{mnR} = \frac{j}{60} V_n \tag{112}$$

$$\sum_{\substack{m=1\\m\neq n}}^{N} A_m^{(1)} \psi'^{(2)}_{mnR} + \sum_{m=1}^{N} A_m^{(2)} \psi'^{(3)}_{mnR} = \frac{j}{60} U_n \tag{113}$$

$$\sum_{m=1}^{N} [A_m^{(1)} \psi'^{(1)}_{mnI} + A_m^{(2)} \psi'^{(2)}_{mnI} + A_m^{(3)} \psi'^{(3)}_{mn}] = 0 . \tag{114}$$

In (113),

$$U_n = - j60 \sum_{m=1}^{N} \int_{-h}^{h} I_m(z')K_{mn}(h,z')dz'$$

$$= - j60 \sum_{m=1}^{N} [A_m^{(1)} \psi_{mn}^{(1)}(h) + A_m^{(2)} \psi_{mn}^{(2)}(h) + A_m^{(3)} \psi_{mn}^{(3)}(h)] \tag{115}$$

where

$$\psi_{mn}^{(1)}(h) = \int_{-h}^{h} S_1(z')K_{mn}(h,z')dz' \tag{116}$$

$$\psi_{mn}^{(2)}(h) = \int_{-h}^{h} S_2(z')K_{mn}(h,z')dz' \tag{117}$$

$$\psi_{mn}^{(3)}(h) = \int_{-h}^{h} S_3(z')K_{mn}(h,z')dz' . \tag{118}$$

With (115) - (118), we can rewrite (113) as

$$\sum_{m=1}^{N} A_m^{(1)}[(1 - \delta_{mn})\psi'^{(2)}_{mnR} - \psi^{(1)}_{mn}(h)]$$

$$+ \sum_{m=1}^{N} A_m^{(2)}[\psi'^{(3)}_{mnR} - \psi^{(2)}_{mn}(h)] = \sum_{m=1}^{N} A_m^{(3)}\psi^{(3)}_{mn}(h) \qquad (119)$$

where δ_{mn} is a Kronecker delta. For an N-element array, $n=1,2,\ldots,N$, and each of (112), (119), and (114) represents N simultaneous equations. It is therefore convenient to use a matrix notation.

IV. MATRIX EQUATIONS

Defining N×1 column matrices $[A^{(1)}]$, $[A^{(2)}]$, $[A^{(3)}]$, and $[V]$, we can write (112), (119), and (114) as

$$[\psi'^{(1)}_{mnR}][A^{(1)}] = \frac{1}{60}[V] \tag{120}$$

$$[(1-\delta_{mn})\psi'^{(2)}_{mnR}-\psi^{(1)}_{mn}(h)][A^{(1)}] + [\psi'^{(3)}_{mnR}-\psi^{(2)}_{mn}(h)][A^{(2)}] = [\psi^{(3)}_{mn}(h)][A^{(3)}] \tag{121}$$

$$[\psi'^{(2)}_{mnI}][A^{(1)}]+ [\psi'^{(2)}_{mnI}][A^{(2)}] + [\psi'^{(3)}_{mn}][A^{(3)}] = 0. \tag{122}$$

Equations (120) - (122) can be inverted and rewritten as

$$[A^{(1)}] = [P^{(1)}][V] \tag{123}$$

$$[A^{(2)}] = [P^{(2)}][V] \tag{124}$$

$$[A^{(3)}] = [P^{(3)}][V] , \tag{125}$$

where

$$[P^{(1)}] = \frac{1}{60}[\psi'^{(2)}_{mnR}]^{-1} \tag{126}$$

$$[P^{(2)}] = [\phi^{(b)}_{p}]^{-1}[\phi^{(a)}_{p}][P^{(1)}]$$

$$[P^{(3)}] = -[\psi'^{(3)}_{mn}]^{-1}[\psi'^{(1)}_{mnI}][P^{(1)}] - [\psi'^{(3)}_{mn}]^{-1}[\psi'^{(2)}_{mnI}][P^{(2)}] \tag{127}$$

and

$$[\phi^{(b)}_{p}] = [\psi^{(3)}_{mn}(h)][\psi'^{(3)}_{mn}]^{-1}[\psi'^{(2)}_{mnI}] + [\psi'^{(3)}_{mnR} - \psi^{(2)}_{mn}(h)] \tag{128}$$

$$[\phi^{(a)}_{p}] = -[(1-\delta_{mn})\psi'^{(2)}_{mnR} - \psi^{(1)}_{mn}(h)] - [\psi^{(3)}_{mn}(h)][\psi'^{(3)}_{mn}]^{-1}[\psi'^{(1)}_{mnI}] \tag{129}$$

The current distribution in the mth dipole in (88) can also be extended to the N-element array and written in a matrix form.

$$[I] = S_1(z)[A^{(1)}] + S_2(z)[A^{(2)}] + S_3(z)[A^{(3)}]. \tag{130}$$

Combining (123) - (125) with (130), we obtain

$$[I] = [Y][V] \tag{131}$$

where the mnth element of the admittance matrix [Y] is

$$Y_{mn}(z) = S_1(z)P_{mn}^{(1)} + S_2(z)P_{mn}^{(2)} + S_3(z)P_{mn}^{(3)} . \tag{132}$$

With (131), we are finally ready to attack the optimization problem. But, before we do this, we shall take care of the case for $\beta h = \pi/2$ (half-wave dipoles) which would make (109) - (111) indeterminate.

V. HALF-WAVE DIPOLES

It is obvious that, when $\beta h = \pi/2$, we could not have proceeded from (85) as we did in Section III. We must start from the basic integral equation (79). When z is set to equal h, (83) becomes

$$\sum_{m=1}^{N} \int_{-h}^{h} I_m(z')K'_{mn}(h,z')dz' = -\frac{1}{60} V_n . \tag{133}$$

Subtracting (133) from (79), we get

$$\sum_{m=1}^{N} \int_{-h}^{h} I_m(z')K'_{mn}(z,z')dz' = -\frac{1}{30} [C_n \cos \beta z - \frac{V_n}{2} (\sin \beta|z| - 1)], \tag{134}$$

where $K'_{mn}(z,z')$ has been defined in (86). C_n can be found from (134) by setting $z = 0$ in (79).

$$C_n = j30 \sum_{m=1}^{N} \int_{-h}^{h} I_m(z')K_{mn}(0,z')dz' \tag{135}$$

Combining (134) and (135), we obtain

$$\sum_{m=1}^{N} \int_{-h}^{h} I_m(z')K'_{mn}(z,z')dz'$$

$$= -\frac{1}{60} [V_n(\sin \beta|z| - 1) + U'_n \cos \beta z] , \qquad n=1,2,\dots,N \tag{136}$$

where

$$U'_n = j60 \sum_{m=1}^{N} \int_{-h}^{h} I_m(z')K_{mn}(0,z')dz'. \tag{137}$$

Equations (136), instead of (85), must now be solved.

Thr three-term expansion for current distribution in (88) - (91) must also be modified. We write

$$I_m(z) = \sum_{k=1}^{3} B_m^{(k)} S_k'(z) \tag{138}$$

with

$$S_1'(z) = \sin \beta|z| - 1 \tag{139}$$

$$S_2'(z) = \cos \beta z \tag{140}$$

$$S_3'(z) = \cos \frac{1}{2} \beta z - \cos \frac{\pi}{4} \tag{141}$$

Following the same procedure outlined in Section III, we obtain, instead of (112) - 114),

$$\sum_{m=1}^{N} B_m^{(1)} X'^{(1)}_{mnR} = - \frac{1}{60} V_n \tag{142}$$

$$\sum_{\substack{m=1 \\ m\neq n}}^{N} B_m^{(1)} X'^{(2)}_{mnR} + \sum_{m=1}^{N} B_m^{(2)} X'^{(3)}_{mnR} = - \frac{1}{60} U_n' \tag{143}$$

$$\sum_{m=1}^{N} [B_m^{(1)} X'^{(1)}_{mnI} + B_m^{(2)} X'^{(2)}_{mnI} - B_m^{(3)} X'^{(3)}_{mn}] = 0 \tag{144}$$

In (143)

$$X'^{(1)}_{mnR} = - \int_{-h}^{h} S_1'(z')[\frac{\cos \beta R_1}{R_1} - \frac{\cos \beta R_2}{R_2}]dz' \tag{145}$$

$$X'^{(2)}_{mnR} = \int_{-h}^{h} S_1'(z')[\frac{\cos \beta R_1}{R_1} - \frac{\cos \beta R_2}{R_2}]dz' \tag{146}$$

$$\chi'^{(3)}_{mnR} = \int_{-h}^{h} S'_2(z')\left[\frac{\cos \beta R_1}{R_1} - \frac{\cos \beta R_2}{R_2}\right]dz' \tag{147}$$

$$\chi'^{(1)}_{mnI} = \frac{-1}{1 - \cos \pi/4} \int_{-h}^{h} S'_1(z')\left[\frac{\sin \beta R_1}{R_1} - \frac{\sin \beta R_2}{R_2}\right]dz' \tag{148}$$

$$\chi'^{(2)}_{mnI} = \frac{-1}{1 - \cos \pi/4} \int_{-h}^{h} S'_2(z')\left[\frac{\sin \beta R_1}{R_1} - \frac{\sin \beta R_2}{R_2}\right]dz'$$

$$\chi'^{(3)}_{mn} = \frac{1}{1 - \cos \pi/4} \int_{-h}^{h} S'_3(z')\left[\frac{e^{-j\beta R_1}}{R_1} - \frac{e^{-j\beta R_2}}{R_2}\right]dz' \quad , \tag{150}$$

where R_1 and R_2 have been defined in (104) and (105). Also,

$$U'_n = j60 \sum_{m=1}^{N} \int_{-h}^{h} I_m(z')K_{mn}(0,z')dz'$$

$$= j60 \sum_{m=1}^{N} \left[B^{(1)}_m \chi^{(1)}_{mn}(0) + B^{(2)}_m \chi^{(2)}_{mn}(0) + B^{(3)}_m(0)\chi^{(3)}_{mn}(0)\right] \tag{151}$$

where

$$\chi^{(1)}_{mn}(0) = \int_{-h}^{h} S'_1(z')K_{mn}(0,z')dz' \tag{152}$$

$$\chi^{(2)}_{mn}(0) = \int_{-h}^{h} S'_2(z')K_{mn}(0,z')dz' \tag{153}$$

$$\chi^{(3)}_{mn}(0) = \int_{-h}^{h} S'_3(z')K_{mn}(0,z')dz' \tag{154}$$

In matrix form, (142) - (144) can be solved to give

$$[B^{(1)}] = [Q^{(1)}][V] \tag{155}$$

$$[B^{(2)}] = [Q^{(2)}][V] \tag{156}$$

$$[B^{(3)}] = [Q^{(3)}][V] \ , \tag{157}$$

where

$$[Q^{(1)}] = -\frac{j}{60}\,[\chi'^{(1)}_{mnR}]^{-1} \tag{158}$$

$$[Q^{(2)}] = [\Phi^{(b)}_Q]^{-1}[\Phi^{(a)}_Q][Q^{(1)}] \tag{159}$$

$$[Q^{(3)}] = -\,[\chi'^{(3)}_{mn}]^{-1}[\chi'^{(1)}_{mnI}][Q^{(1)}] - [\chi'^{(3)}_{mn}]^{-1}[\chi'^{(2)}_{mn}][Q^{(2)}] \tag{160}$$

and

$$[\Phi^{(b)}_Q] = [\chi^{(3)}_{mn}(0)][\chi'^{(3)}_{mn}]^{-1}[\chi'^{(2)}_{mnI}] + [\chi'^{(3)}_{mnR} - \chi^{(2)}_{mn}(0)] \tag{161}$$

$$[\Phi^{(2)}_Q] = -\,[(1-\delta_{mn})\chi'^{(2)}_{mnR} - \chi^{(1)}_{mn}(0)] - [\chi^{(3)}_{mn}(0)][\chi'^{(3)}_{mn}]^{-1}[\chi'^{(1)}_{mnI}] \tag{162}$$

Combining (149) - (151) with (138), we have

$$[I] = [Y'][V] \ , \tag{163}$$

where

$$Y'_{mn}(z) \quad S'_1(z)Q^{(1)}_{mn} + S'_2(z)Q^{(2)}_{mn} + S'_3(z)Q^{(3)}_{mn} \tag{164}$$

Equations (163) and (164) for half-wave dipoles correspond to (131) and (132) respectively.

VI. MAXIMIZATION OF DIRECTIVITY

The far-zone electric field of an array of N parallel, z-directed, center-fed dipoles is

$$E(\theta,\phi) = \frac{j\omega\mu}{4\pi r} e^{-j\beta r} \sum_{n=1}^{N} e^{j\beta \bar{r}_n \cdot \hat{u}} \int_{-h}^{h} I_n(z') e^{j\beta z' \cos\theta} \sin\theta \, dz' \, , \quad (165)$$

where β is the phase constant, $\bar{r}_n$ is the vector from the origin to the center of the nth dipole, and $\hat{u}$ is the unit vector from the origin to the observation point. Equation (165) can be rearranged and written in a matrix notation.

$$E(\theta,\phi) = \frac{e^{-j\beta r}}{r} \int_{-h}^{h} [H]^{\dagger}[I] \, dz' \, , \quad (166)$$

where both [H] and [I] are N×1 column matrices. The typical elements of [H] is H_n:

$$H_n = \frac{-j\omega\mu \sin\theta}{4\pi} e^{-j\beta \bar{r}_n \cdot \hat{u}} \left(e^{-j\beta z' \cos\theta}\right) \, . \quad (167)$$

Substituting (131) in (166), we obtain

$$E(\theta,\phi) = \frac{e^{-j\beta r}}{r} \left\{ \int_{-h}^{h} [H]^{\dagger}[Y] dz' \right\} [V]$$

$$= \frac{e^{-j\beta r}}{r} [M]^{\dagger}[V] \, , \quad (168)$$

where

$$[M]^{\dagger} = \int_{-h}^{h} [H]^{\dagger}[Y] \, dz' \quad (169)$$

is a $1{\times}N$ row matrix. Of course, (163), instead of (131), would be used if $\beta h = \pi/2$. The only effect would be to change [Y] in (169) to [Y'].

Now, the array directivity is, from (65),

$$D(\theta_o,\phi_o) = \frac{r^2|E(\theta_o,\phi_o)|^2}{60\ P_{in}} \tag{170}$$

where P_{in} is the time-average input power to the array.

$$\begin{aligned} P_{in} &= \frac{1}{2}\,\mathrm{Re}\,\{[I]^{\dagger}_{z=0}\cdot\ [V]\} \\ &= \frac{1}{2}\,\mathrm{Re}\,\{[V]\ [Y]^{\dagger}_{z=0}\cdot[V]\} \\ &= \frac{1}{2}\,[V]^{\dagger}[Y_R][V]. \end{aligned} \tag{171}$$

The elements of $[Y_R]$ are the real part of those of the driving point admittance matrix $[Y]_{z=0}$. With (168) and (171), $D(\theta_o,\phi_o)$ in (170) becomes

$$D(\theta_o,\phi_o) = \frac{[V]^{\dagger}[M_o][M_o]^{\dagger}[V]}{30\ [V]^{\dagger}[Y_R]\ [V]}$$

where $[M_o]$ is [M] in (169) evaluated for the direction (θ_o,ϕ_o). Equation (172), as (7), is a ratio of Hermitian forms, and Theorem 1 in Section III, Part (A), can be used to find the maximum directivity $D_M(\theta_o,\phi_o)$, and the required voltage excitations [V]. We have

$$D_M = \frac{1}{30}\,[M_o]^{\dagger}[Y_R]^{-1}[M_o] \tag{173}$$

and

$$[V]_M = [Y_R]^{-1}[M_o]\ . \tag{174}$$

The optimization problem is now completely solved. The required matrix $[Y_R]$ is obtained easily from the admittance matrix $[Y]$ (or $[Y']$ if $\beta h = \pi/2$), whose formulation has been developed in the previous sections. We note that $[Y_R]$ is of a dimension $N \times N$ for an N-element array irrespective of the length of the dipoles.

VII. NUMERICAL EXAMPLE

The integral-equation approach formulated above for optimizing arrays with mutual coupling is applied to a four-element circular array for which some important results on directivity optimization have been obtained by using the method of moments. Although this approach applies to larger arrays, it was thought advisable to check with some known results first in view of the rather involved analytical process. Once the formulation and the results have been verified for the four-element array, the extension to larger arrays is straightforward and needs only slight changes in the computer program.

The array has four parallel, z-directed, dipoles, each of diameter 2a and length 2h, uniformly spaced around a circle of diameter d. The coordinate system is selected such that one dipole coincides with the z-axis, a second one lying in the xz-plane, and a third one lying in the yz-plane. The following parameters are chosen:

$$2a/\lambda = 0.0025$$

$$2h/\lambda = 0.36 \quad (h/a = 144)$$

$$d/\lambda = 0.61 .$$

The directivity of this array in the principal H-plane ($\theta = 90°$) is maximized for each value of ϕ by adjusting the amplitudes and the phases of the excitation voltages V_n, $n=1,2,3,4$. The results are plotted as the curve in Fig. 8, which coincides almost exactly with that obtained by Cummins [23],

who used the method of moments and the method of subsections described in Section VII, Part (A). The Fortran program used for computing the maximum directivity in Fig. 8 is appended to show its relative simplicity. In spite of the complexity of the formalism, the cost of computing the entire curve in Fig. 8 on an IBM 360/50 computer was only about five dollars.

VIII. CONCLUSION

The obvious advantage of the integral-equation approach in solving the directivity maximization problem for an array of dipole antennas, as compared to the method of moments using subsections, is the S-fold reduction in the order of the matrix to be inverted, where S is the number of subsections for each dipole. This results in a saving in computer time (cost) by a factor of S^3, in addition to relaxing the requirements on memory capacity.* Besides the directivity, quantities such as the current distribution on each dipole, the self and mutual impedances, the radiation pattern, etc., can all be calculated without difficulty.

For arrays with many elements (N very large), practical computing difficulties will arise even if the antennas are not divided into subsections. In such cases, other techniques are needed to simplify the computing procedure. Because of the existence of rotational symmetry in a uniformly spaced circular array, it is possible to circumvent the necessity of inverting any matrix by the introduction of a rotational operator [17]. Hence, the number of elements in a uniform circular array, no matter how large, represents no real constraint on the feasibility of obtaining numerical solutions. On the other hand, linear arrays possess no circular symmetry; thus the technique of using a rotational operator does not apply and other methods must be sought when N is very large. Special methods for handling pattern synthesis and performance optimization of very large arrays constitute an important area for further research.

---

*For the particular example in Section VII, symmetry about the plane (xy-plane) bisecting the dipoles reduces the effective order of the matrix handled by the method of moments by a factor of 2. Symmetry property about the plane containing the diametrically opposite elements simplifies the computation for both the moment method and the integral-equation method.

APPENDIX TO PART (B)

FORTRAN PROGRAM FOR COMPUTING DIRECTIVITY OF A FOUR-ELEMENT CIRCULAR ARRAY AS A FUNCTION OF AZIMUTH ANGLE

```
      SUBROUTINE CGMP(AB,BC,RC,N,M,L)
      COMPLEX*8 AB(N,L),BC(L,M),RC(N,M)
      DO 10 I=1,N
      DO 10 J=1,M
      RC(I,J)=(0.,0.)
      DO 10 K=1,L
   10 RC(I,J)=RC(I,J)+AB(I,K)*BC(K,J)
      RETURN
      END
      SUBROUTINE CAIMV(AB,AR,M)
      COMPLEX*8 AB(M,M),AR(M,M),BI,BM(4,4),S1(4,4),T1(4,4),RN(4,4)
      REAL*4 C1,C2,C3,C4
      BI=(0.,1.)
      DO 106 I=1,M
      DO 106 J=1,M
      IF(I-J)104,105,104
  105 BM(I,J)=(0.,0.)
      DO 103 KI=1,M
      C1=2.*1.0000*FLOAT((KI-1)*(I-1))/FLOAT(M)*3.141592653
  103 BM(I,J)=BM(I,J)+AB(1,KI)*(COS(C1)+BI*SIN(C1))
      BM(I,J)=1./BM(I,J)
      GO TO 106
  104 BM(I,J)=(0.,0.)
  106 CONTINUE
      DO 1 I=1,M
      DO 1 J=1,M
      C3=FLOAT(M)
      C4=SQRT(C3)
  107 C2=2.*1.0000*FLOAT((J-1)*(I-1))/FLOAT(M)*3.141592653
      S1(I,J)=(COS(C2)+BI*SIN(C2))/C4
      T1(J,I)=(COS(C2)-BI*SIN(C2))/C4
    1 CONTINUE
      DO 10 I=1,M
      DO 10 J=1,M
      RN(I,J)=(0.,0.)
      DO 10 K=1,M
   10 RN(I,J)=RN(I,J)+S1(I,K)*BM(K,J)
      DO 11 I=1,M
      DO 11 J=1,M
      AR(I,J)=(0.,0.)
      DO 11 K=1,M
   11 AR(I,J)=AR(I,J)+RN(I,K)*T1(K,J)
      RETURN
      END
      INTEGER IABS,IP
      COMPLEX*8 AA(4,4),AB(4,4),AC(4,4),AD(4,4),AE(4,4),
     1AF(4,4),BA(4,4),BB(4,4),BC(4,4)
      COMPLEX*8 P1(4,4),PB(4,4),PC(4,4),PD(4,4),PE(4,4),T1(4,4),T2(4,4)
     1,PF(4,4),P2(4,4),PG(4,4),BI
     1,BH(4,4),BI(4,4),BJ(4,4),BK(4,4),V,P3(4,4)
      REAL*4 [illegible],AI,[illegible](5,1),F(7),G(7),[illegible](5,7),SS(5,7),S,T,VV(4),
     1PHAS(4),[illegible]PO(4),GAII,ABS,SS(5,5),TT(5,5)
      COMPLEX*8 YY(4,4),D6,D7,DD(4),DE(4),YI(4,4),VI,V(4),GAIN
      BI=(0.,1.)
      PLENT=0.19
      P=0.61
      H(3,1)=P
      H(2,1)=P*0.707
      H(1,1)=PLENT/144.
      DO 7 KI=1,3
```

NOT REPRODUCIBLE

```
      DO 7 K2=1,7
      DO 1 I=1,4
      N=10*2**(I-1)
      H=2.*HLENT/FLOAT(N)
      N1=N-1
      T=0.
      S=0.
      DO 2 II=1,N1
      ARG3=2.*3.1416*HLENT
      B8=SIN(0.5*ARG3)
      B7=COS(0.5*ARG3)
      B6=SIN(ARG3)
      B5=COS(ARG3)
      Z=-HLENT+H*FLOAT(II)
      R1=Z**2+D(K1,1)**2
      R2=(HLENT-Z)**2+D(K1,1)**2
      R3=(HLENT-0.25-Z)**2+D(K1,1)**2
      ARG4=2.*3.1416*SQRT(R3)
      B12=COS(ARG4)
      ARG1=2.*3.1416*SQRT(R1)
      ARG2=2.*3.1416*SQRT(R2)
      Z1=ABS(Z)
      B11=COS(3.1416*Z)
      B10=COS(2.*3.1416*Z)
      ARG5=HLENT-Z1
      B9=SIN(2.*3.1416*ARG5)
      B4=SIN(ARG2)
      B3=COS(ARG2)
      B2=SIN(ARG1)
      B1=COS(ARG1)
      F(1)=B9*(B1/ARG1-B3/ARG2)
      F(2)=(B10-B5)*(B1/ARG1-B3/ARG2)
      F(3)=B9*(B2/ARG1-B4/ARG2)
      F(4)=(B10-B5)*(B2/ARG1-B4/ARG2)
      F(5)=(B11-B7)*(B1/ARG1-B3/ARG2)
      F(6)=(B11-B7)*(B2/ARG1-B4/ARG2)
      F(7)=B9*(B12/ARG4-B3/ARG2)
      G(6)=(B11-B7)*B4/ARG2
      G(5)=(B11-B7)*B3/ARG2
      G(4)=(B10-B5)*B4/ARG2
      G(3)=(B10-B5)*B3/ARG2
      G(2)=B9*B4/ARG2
      G(1)=B9*B3/ARG2
      G(7)=0.
      S=S+G(K2)
    2 T=T+F(K2)
      SS(I,1)=S*H
    1 TT(I,1)=T*H
      DO 3 K=1,3
      J=4-K
      DO 3 I=1,J
      SS(I,K+1)=(FLOAT(4**K)*SS(I+1,K)-SS(I,K))/(FLOAT(4**K-1))
    3 TT(I,K+1)=(FLOAT(4**K)*TT(I+1,K)-TT(I,K))/(FLOAT(4**K-1))
      TR(K1,K2)=TT(1,4)*6.2832
    7 SR(K1,K2)=SS(1,4)*6.2832
      DO 52 I=1,4
      DO 52 J=1,4
      IQ=IABS(I-J)
      IF(IQ-1)53,54,54
   53 L=1
```

```
      PA(I,J)=-(SR(L,1)-RI*SR(L,2))/R5
      GO TO 57
   54 IF(IO-2)55,56,55
   55 L=2
      GO TO 57
   56 L=3
   57 AA(I,J)=-RI*60.*R5*TR(L,1)/R6
      AB(I,J)=TR(L,1)/(1.-R5)+RI*0.
      AC(I,J)=TR(L,2)/(1.-R5)+RI*0.
      AD(I,J)=-RI*TR(L,3)/(1.-R7)
      AE(I,J)=-RI*TR(L,4)/(1.-R7)
      AF(I,J)=(TR(L,5)-RI*TR(L,6))/(1.-R7)
      BB(I,J)=AC(I,J)-(SR(L,3)-RI*SR(L,4))/R5
      BC(I,J)=(SR(L,5)-RI*SR(L,6))/R5
      IF(I-J)59,58,59
   58 BA(I,J)=BA(I,J)-(SR(L,1)-RI*SR(L,2))/R5
   59 WRITE(3,1101)I,J,AA(I,J),R5,R6
 1101 FORMAT(1H0,'AA(',I3,I3,')=',2E12.5/1X,'R5=',E12.5,'R6=',E12.5)
      DO 20 I=1,4
      DO 20 J=1,4
      IF(I-J)21,22,21
   22 P1(I,J)=RI*R6/(60.*R5*TR(1,1))
      GO TO 20
   21 P1(I,J)=(0.,0.)
   20 CONTINUE
      CALL CAINV(AE,AP,4)
      CALL CGMP(BC,AP,RC,4,4,4)
      CALL CGMP(RC,AE,RD,4,4,4)
      DO 200 I=1,4
      DO 200 J=1,4
      Q1(I,J)=RD(I,J)+BA(I,J)
  200 CONTINUE
      CALL CGMP(RC,AD,RE,4,4,4)
      DO 201 I=1,4
      DO 201 J=1,4
  201 Q2(I,J)=BB(I,J)-RE(I,J)
      CALL CAINV(Q1,RF,4)
      CALL CGMP(RF,Q2,RG,4,4,4)
      CALL CGMP(RG,P1,P2,4,4,4)
      CALL CGMP(AB,AD,RH,4,4,4)
      CALL CGMP(RH,P1,RI,4,4,4)
      CALL CGMP(
      CALL CGMP(RJ,P2,RK,4,4,4)
      DO 300 I=1,4
      DO 300 J=1,4
      PA(I,J)=-RI(I,J)-RK(I,J)
  300 CONTINUE
      DO 400 I=1,4
      DO 400 J=1,4
      Y=R6*P1(I,J)+(1.-R5)*P2(I,J)+(1.-R7)*P3(I,J)
      YZ=REAL(Y)
      YY(I,J)=YZ+RI*0.
  400 WRITE(3,4001)I,J,YY(I,J)
 4001 FORMAT(1H0,'YY(',I3,I3,')=',2E12.5)
      RK00=-3.1416/40.
      DO 803 II=1,20
      RK00=RK00+3.1416/40.
      RKP0(1)=0.
      RKP0(2)=2.*3.1416*R*0.707*COS(RK00)
      RKP0(3)=2.*3.1416*R*0.707*(COS(RK00)+SIN(RK00))
```

```
      PKPO(4)=2.*3.1416*P*0.707*SIN(PK00)
      DO 701 I=1,4
      D6=(0.,0.)
      DO 702 J=1,4
      D3=PKPO(J)
  702 D6=D6+(COS(D3)+[illegible]*SIN(D3))*(P1(J,I)**2.*(1.-[illegible])+P2(J,I)**2.*(A0-
     1ARG3*A5)+P3(J,I)**2.*(2.*A8-ARG3*A7))
      D7=A1*30.*D6
      DD(I)=C[illegible](D7)
      DF(I)=D7
  701 WRITE(3,7000)I,DD(I),I,DF(I)
 7000 FORMAT([illegible]'DF(',I3,')=',2E12.5/1X,'DD(',I3,')=',2E12.5)
      CALL CAL[illegible]V(YY,YI,4)
      DO 800 I=1,4
      V1=(0.,0.)
      DO 801 J=1,4
  801 V1=V1+YI(I,J)*[illegible](J)
      V(I)=V1
      VV(I)=CABS(V1)
      V3=REAL(V1)
      V4=AIMAG(V1)
      V5=V4/V3
      PHAS(I)=-ATAN(V5)*180./3.1416
  800 WRITE(3,8000)I,V(I),I,VV(I),I,PHAS(I)
 8000 FORMAT('0','V(',I3,')=',2E12.5/2X,'VV(',I3,')=',E12.5/2X,'PHAS(',
     1I3,')=',E12.5)
      GAIN=(0.,0.)
      DO 802 I=1,4
  802 GAIN=GAIN+DE(I)*V(I)/30.
      GAI1=CABS(GAIN)
  803 WRITE(3,8001)GAI1
 8001 FORMAT('1','GAIN=',E12.5)
      STOP
      END
```

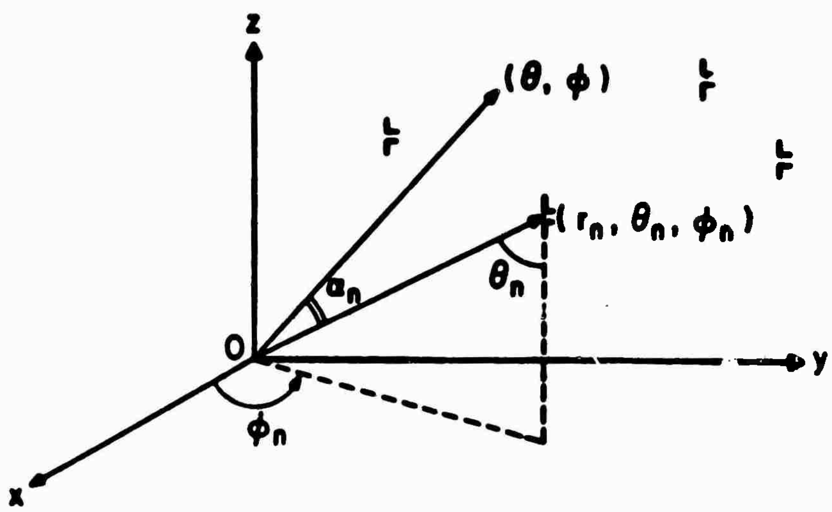

Fig. 1 - Reference coordinates of an arbitrary array.

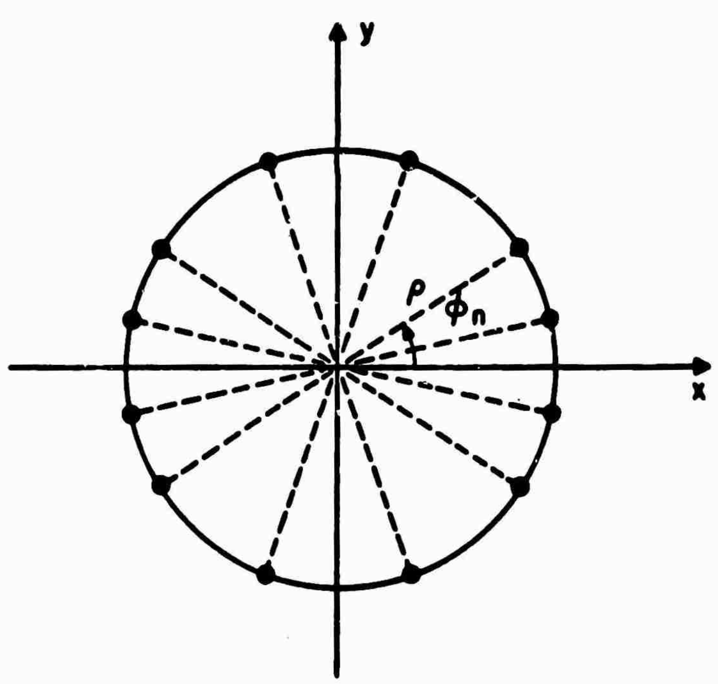

Fig. 2 - A circular array.

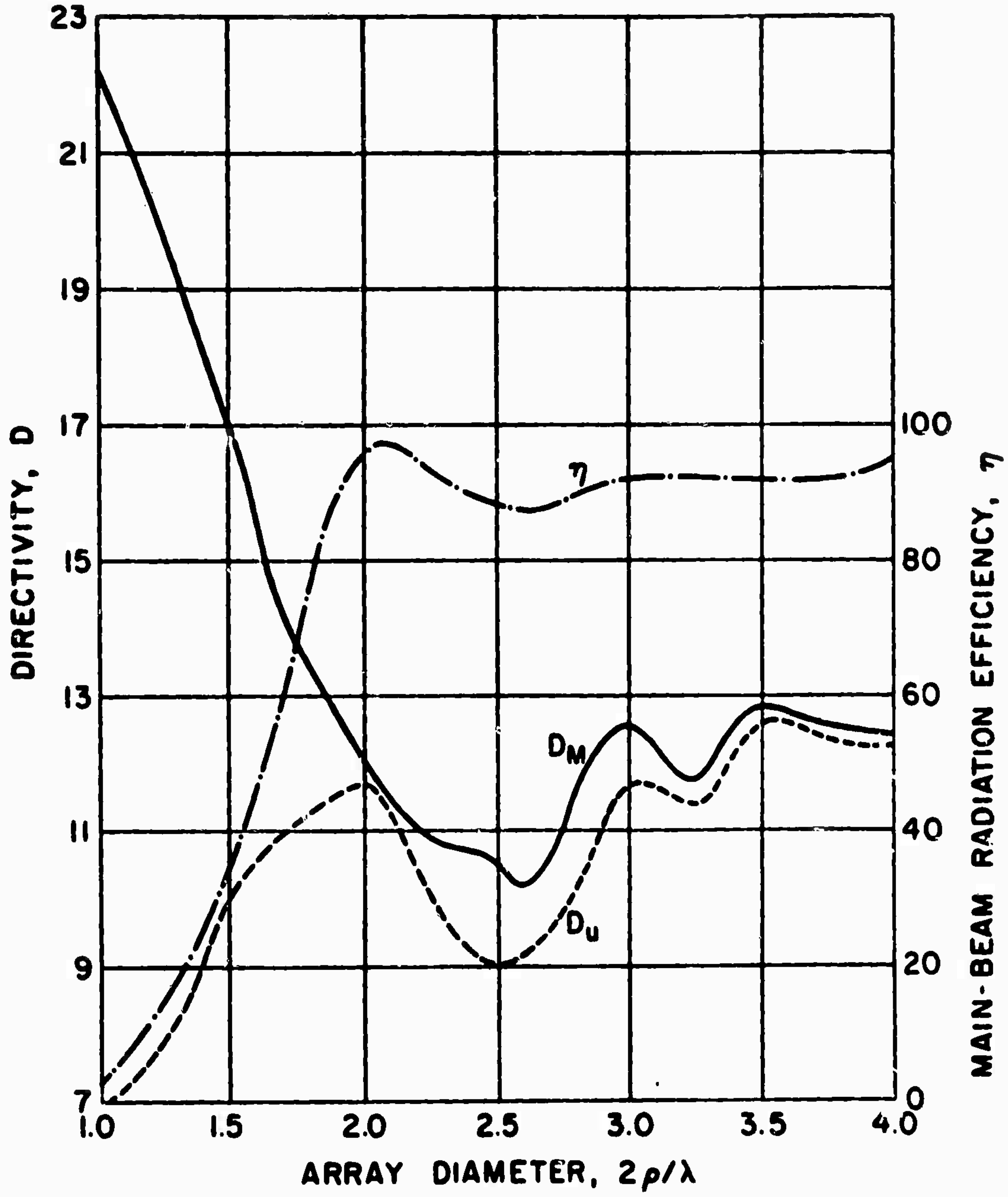

Fig. 3 - Directivity and main-beam radiation efficiency for a circular array with isotropic elements ($N=12$, $\theta_o=90°$, $\phi_o=0°$).

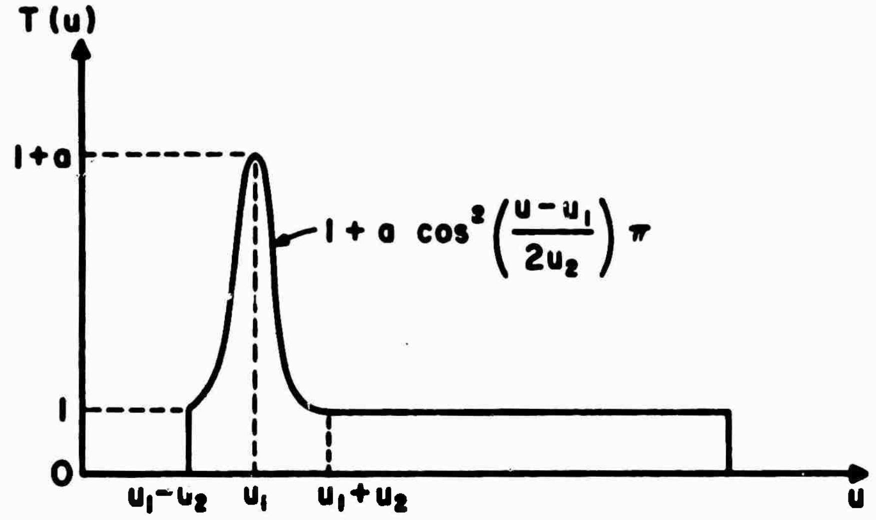

Fig. 4 - Spatial distribution of noise power; $u = (2\pi d/\lambda)\sin\theta$.

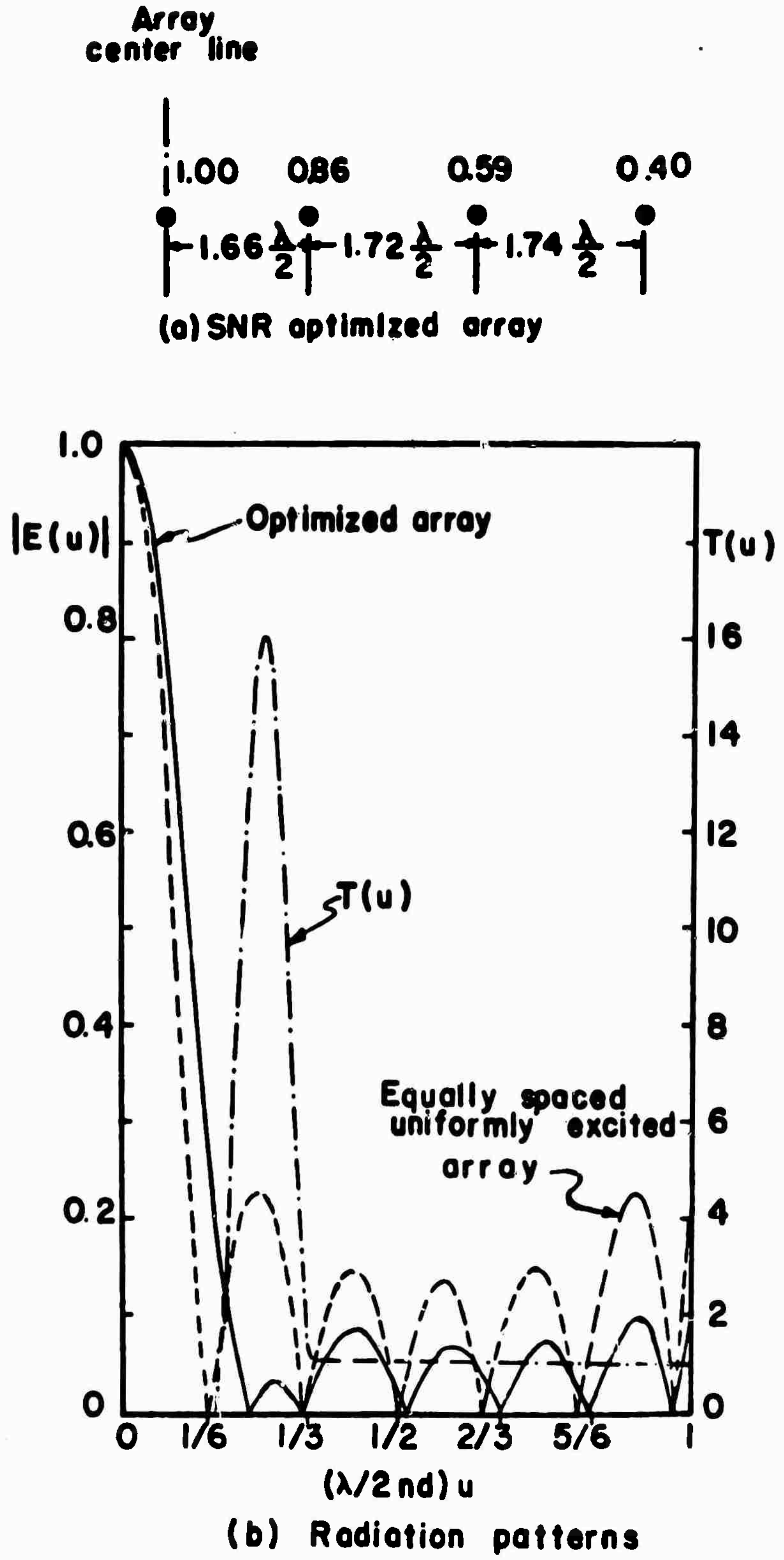

Fig. 5 - SNR optimization for 7-element broadside array.

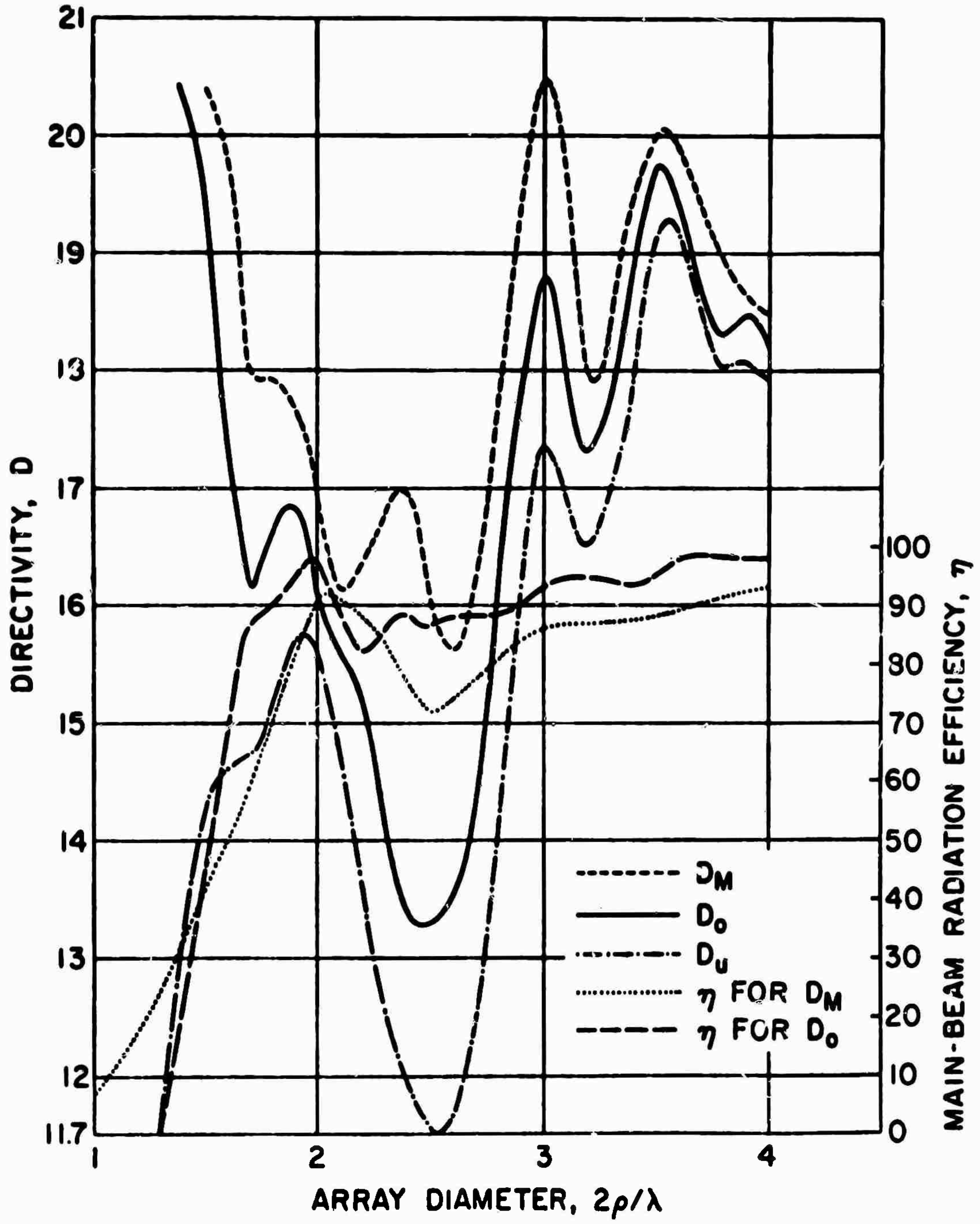

Fig. 6 - Optimum directivity and main-beam radiation efficiency for a circular array with dipole elements ($N=12$, $\theta_o=90°$, $\phi_o=0°$).

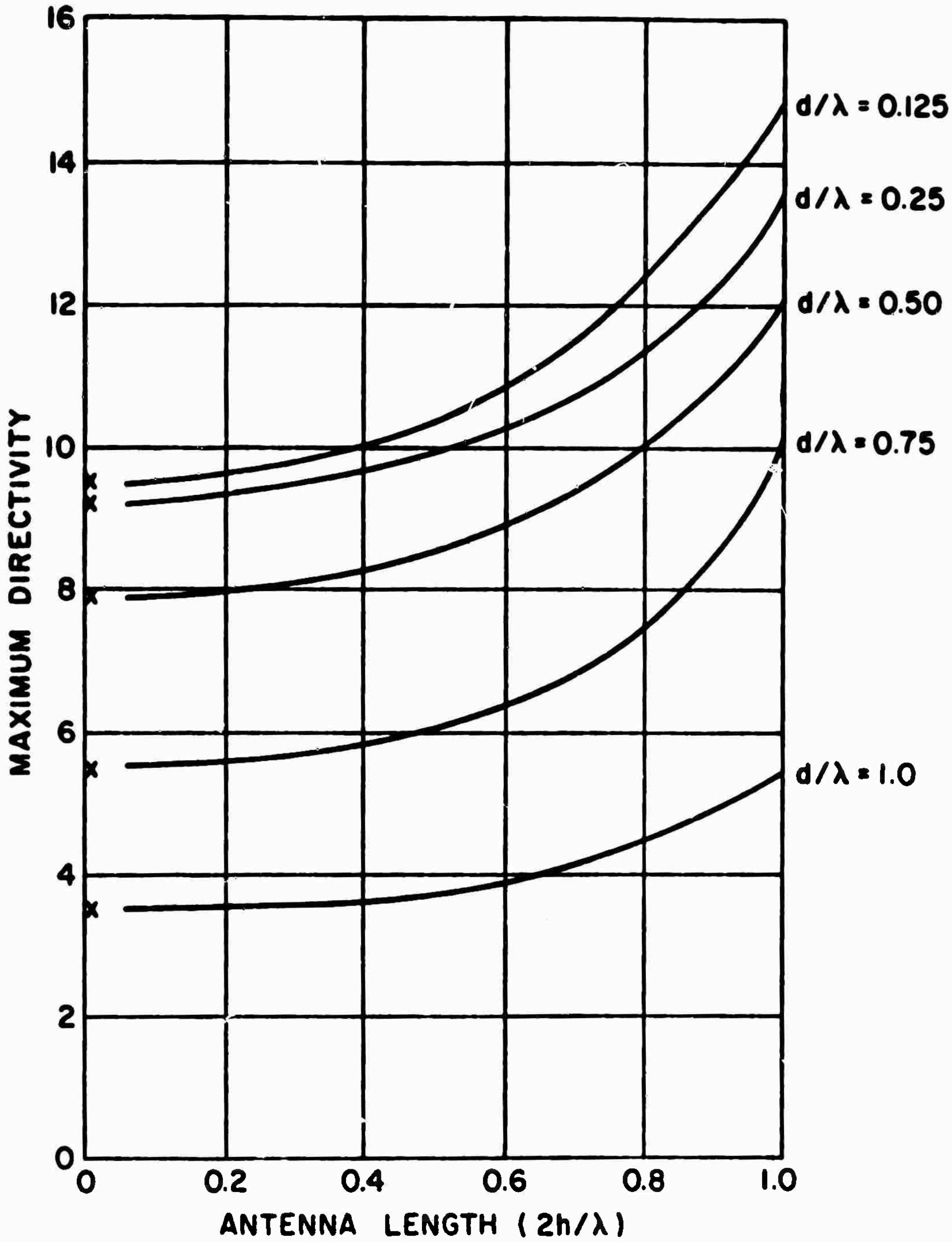

Fig. 7 - Maximum directivity of a 4-element circular array in direction $\theta=90°$ and $\phi=45°$.

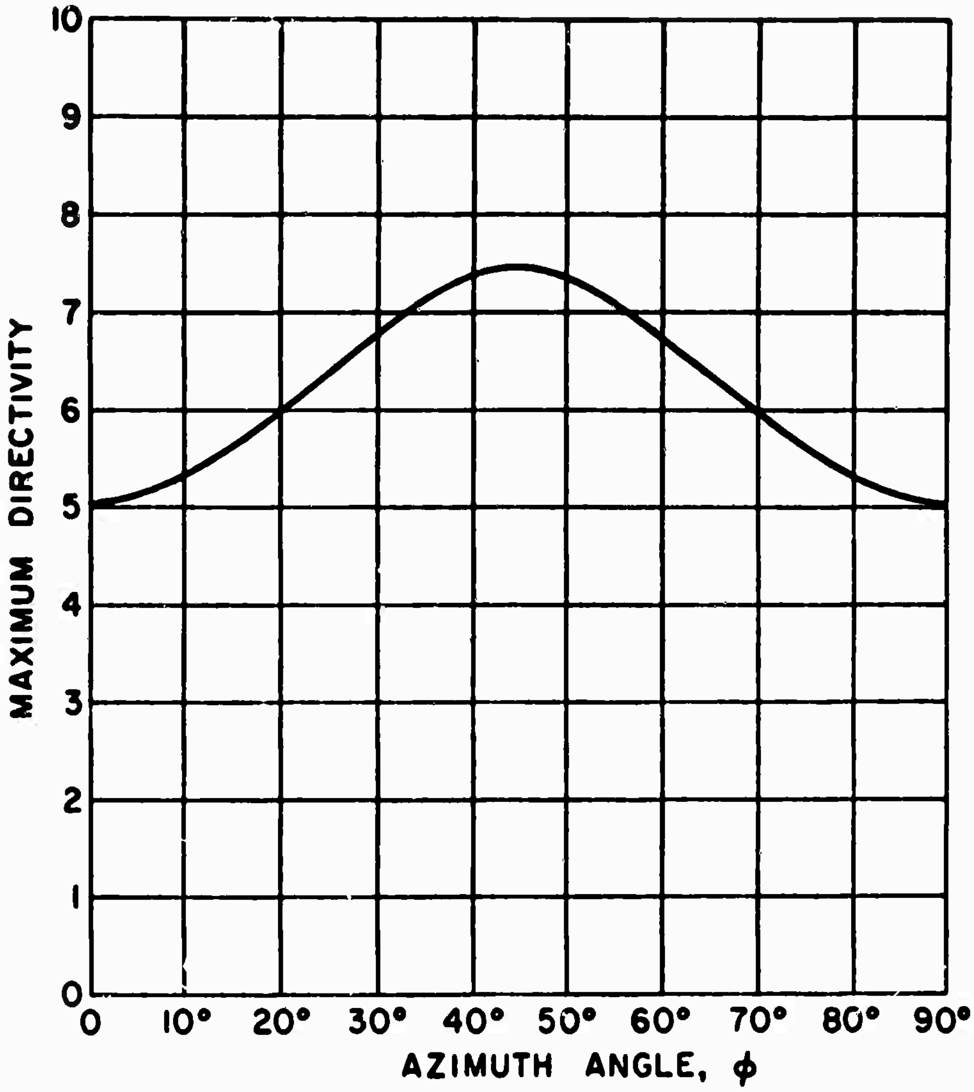

Fig. 8 - Maximum directivity of a 4-element circular array as a function of azimuth angle.

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